

جامعة الفرات الأوسط
المعهد التقني النجف
تقنيات الأجهزة الطبية



الرياضيات
المرحلة الأولى
2026-2025
م.م فاطمة حيدر حسن
fatimahhayder.inj@atu.edu.iq

الهدف العام: مساعدة الطالب على فهم القوانين والمسائل الرياضية اللازمة وتطبيقاتها الأساسية والمتقدمة.

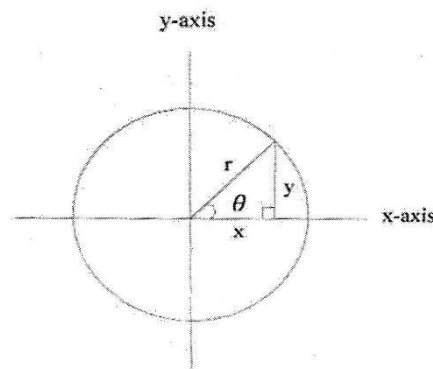
الأسبوع	تفاصيل المفردات
2-1	الدوال المثلثية: التعاريف الأساسية، المتطابقات المثلثية، إشارات الأرباع، ودوال المجموع والفرق.
3-2	الدوال (Domain & Range) تحليل الدوال: مجال ومدى الدوال الزوجية والفردية وتناظرها.
5-4	معادلات المستقيم والدائرة: ميل المستقيم، أشكال معادلة الخط المستقيم، المعادلة القياسية والعامة للدائرة.
7-6	غايات الدوال من طرف واحد، قوانين الغايات، وحل (Limits) الغايات عند اللانهاية.
9-8	شروط استمرارية الدالة عند نقطة، أنواع (Continuity) الاستمرارية عدم الاستمرارية، وإيجاد الثوابت.
11-10	قواعد الاشتقاق الأساسية: المشتقة والمماس، قواعد اشتقاق القوة، الضرب، القسمة، والدوال المثلثية.
13-12	والاشتقاق (Chain Rule) طرق الاشتقاق المتقدمة: قاعدة السلسلة (Implicit Differentiation) الضمني.
15-14	تطبيقات المشتقة 1: (الدوال التزايدية والتناقصية، ورسم المنحنيات باستعمال المشتقة الأولى والثانية).
17-16	تطبيقات المشتقة 2: (الحركة والديناميكا) الإزاحة، السرعة، والتعجيل).
19-18	التكامل غير المحدد والمحدد: القواعد الأساسية للتكامل، خواص التكامل المحدد وتكامل الدوال المتناظرة.
21-20	(u- تقنيات التكامل وتطبيقاته: تكامل الدوال المثلثية، طريقة التعويض

substitution)، وإيجاد معادلة المنحنى.	
العمليات على المصفوفات (الجمع، الضرب، (Matrices) المصفوفات والمدور)، وأنواع المصفوفات الخاصة	23-22
المحددات وأنظمة المعادلات: حساب المحددات، طريقة كرامر لحل أنظمة المعادلات الخطية (Cramer's Rule)	27-24
الدوال اللوغاريتمية: خواص اللوغاريتم الطبيعي، اشتقاق وتكامل الدوال اللوغاريتمية والاشتقاق اللوغاريتمي	29-28
ورسمها، واشتقاق وتكامل (e^x) الدوال الأسية: خواص الدالة الأسية الدوال الأسية	30-29

1-1 Trigonometric Functions: الدوال المثلثية

Given central circle with a radius (r), The point P(x,y) could be any point on the circle then:

- Sine of $\theta \rightarrow \sin \theta = \frac{y}{r}$
- Cosine of $\theta \rightarrow \cos \theta = \frac{x}{r}$
- Tangent of $\theta \rightarrow \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{y}{x}$
- Secant of $\theta \rightarrow \sec \theta = \frac{1}{\cos \theta} = \frac{r}{x}$
- Cosecant of $\theta \rightarrow \csc \theta = \frac{1}{\sin \theta} = \frac{r}{y}$
- Cotangent of $\theta \rightarrow \cot \theta = \frac{1}{\tan \theta} = \frac{x}{y}$



From above triangle we can say:

$$r^2 = x^2 + y^2 \quad \text{(فيثاغورس)}$$

But: $x = r \cos \theta$ & $y = r \sin \theta$

$$\therefore r^2 = (r \cos \theta)^2 + (r \sin \theta)^2$$

$$r^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta$$

$$r^2 = r^2 (\cos^2 \theta + \sin^2 \theta)$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

Some trigonometric Identities:

- $\sin^2 x + \cos^2 x = 1$ (1)
- $\tan^2 x + 1 = \sec^2 x$ (ناتجة من قسمة المعادلة رقم 1 على $\cos^2 x$)
- $1 + \cot^2 x = \csc^2 x$ (ناتجة من قسمة المعادلة رقم 1 على $\sin^2 x$)
- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$
- $\cos 2x = (1 - \sin^2 x) - \sin^2 x \rightarrow \cos 2x = 1 - 2\sin^2 x$
- $2\sin^2 x = 1 - \cos 2x \rightarrow \therefore \sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- Also, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\sin(x - y) = \sin x \cos y - \cos x \sin y$
- $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- $\cos(x - y) = \cos x \cos y + \sin x \sin y$

$$\text{➤ } \tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\text{➤ } \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$\text{➤ } \sin A \cdot \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

$$\text{➤ } \cos A \cdot \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$$

$$\text{➤ } \sin A \cdot \cos B = \frac{1}{2} [\sin(A - B) + \sin(A + B)]$$

• **Prove that:** $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$\begin{aligned} \text{Right hand side} &= \frac{1}{2}(1 - \cos 2x) \\ &= \frac{1}{2}[1 - (\cos^2 x - \sin^2 x)] \\ &= \frac{1}{2}[1 - (1 - \sin^2 x - \sin^2 x)] \\ &= \frac{1}{2}[1 - 1 + 2\sin^2 x] \end{aligned}$$

$$\sin^2 x = \text{the left side} \Rightarrow \therefore \text{o.K.}$$

قياس بعض الزوايا الخاصة

الربع الثاني (sin)	الربع الأول (sin, cos, tan...)
الربع الثالث (tan)	الربع الرابع (cos)

❖ إضافة أو طرح أي عدد من الدورات لا يغير من قيمة الزوايا (2π دورة واحدة & 4π دورتان & 6π ثلاث دورات و هكذا).

$$360 = 2\pi \text{ \& } 270 = \frac{3\pi}{2} \text{ \& } 180 = \pi \text{ \& } 90 = \frac{\pi}{2} \text{ ❖}$$

◀ الزوايا المكملية (2π & π) : موقع الربع يحدد الإشارة و تبقى الدالة نفسها مثلا دالة ال $\sin$ تبقى $\sin$ كما في الامثلة ادناه :-

$$\begin{array}{ll} \sin(\pi - \theta) = + \sin \theta & \text{الربع الثاني} \\ \sin(\pi + \theta) = - \sin \theta & \text{الربع الثالث} \\ \cos(\pi + \theta) = - \cos \theta & \text{الربع الثالث} \end{array} \quad \begin{array}{ll} \cos(\pi + \theta) = - \cos \theta & \text{الربع الثالث} \\ \tan(\pi - \theta) = - \tan \theta & \text{الربع الثاني} \\ \tan(\pi + \theta) = + \tan \theta & \text{الربع الثالث} \end{array}$$

$$\left. \begin{array}{l} \sin(2\pi - \theta) = - \sin \theta \\ \tan(2\pi - \theta) = - \tan \theta \\ \cos(2\pi - \theta) = + \cos \theta \end{array} \right\} \text{زوايا في الربع الرابع}$$

لاحظ انه:-

$$\sin(-\theta) = - \sin \theta , \quad \cos(-\theta) = \cos \theta , \quad \tan(-\theta) = - \tan(\theta)$$

◀ الزوايا المتممة ($\frac{3\pi}{2}$ & $\frac{\pi}{2}$) :: موقع الربع يحدد الإشارة و تقلب الدالة الى متممتها. مثلا دالة ال $\sin$ في الربع الأول تكون موجبة ثم تقلب الى $\cos$.

زوايا في مختلف الارباع	
$\sin(\frac{\pi}{2} + \theta) = + \cos \theta$ الربع الثاني	$\cos(\frac{\pi}{2} + \theta) = - \sin \theta$ الربع الثاني
$\cos(\frac{\pi}{2} + \theta) = - \sin \theta$ الربع الثاني	$\tan(\frac{\pi}{2} + \theta) = - \cot \theta$ الربع الثاني
$\sin(\frac{\pi}{2} - \theta) = + \cos \theta$ الربع الاول	$\tan(\frac{\pi}{2} - \theta) = + \cot \theta$ الربع الاول
$\sin(\frac{3\pi}{2} - \theta) = - \cos \theta$ الربع الثالث	$\sin(\frac{3\pi}{2} + \theta) = - \cos \theta$ الربع الرابع
$\cos(\frac{3\pi}{2} - \theta) = - \sin \theta$ الربع الثالث	$\tan(\frac{3\pi}{2} - \theta) = + \cot \theta$ الربع الثالث
$\cos(\frac{3\pi}{2} + \theta) = + \sin \theta$ الربع الرابع	$\tan(\frac{3\pi}{2} + \theta) = - \cot \theta$ الربع الرابع

✚ Some examples about Trigonometric Functions

- $\sin 10x = 2 \sin 5x \cos 5x$
- $\cos 4x = \cos^2 2x - \sin^2 2x$
- $\cos^2 7x = \frac{1}{2}(1 + \cos 14x)$
- $\sin^2 15 = \frac{1}{2}(1 - \cos 30)$
- $\cos 330 = \cos (270 + 60) = + \sin 60$
or $\cos 330 = \cos (360 - 30) = \cos 30$
- $\sin 150 = \sin(180 - 30) = + \sin 30$

1-2 Functions: الدوال

1-2-1 the Domains and the Ranges:

Domain: هو مجال القيم الحقيقية لـ x (التي تأخذ قيم حقيقية لـ y). و صيغة

المعادلة تكون: Equation $\rightarrow y = f(x)$

Range: هو مجال القيم الحقيقية لـ y (التي تأخذ قيم حقيقية لـ x). و صيغة

المعادلة تكون: Equation $\rightarrow x = f(y)$

Domain and Range are divided in to bounded and Infinite Intervals:

Bounded Intervals الفترات المحددة		Infinite Intervals الفترات الغير المحددة	
1	$a < x < b$ $a(\text{---})b \rightarrow \text{open intervals}$ $D_f: a < x < b \text{ or } (a, b)$	1	$-\infty < x < \infty$ $-\infty \leftarrow 0 \rightarrow \infty$ $D_f: -\infty < x < \infty \text{ or } (-\infty, \infty)$ Or R (all real number)
2	$a \leq x \leq b$ $a[\text{---}]b \rightarrow \text{close interval}$ $D_f: a \leq x \leq b \text{ or } [a, b]$	2	$a < x$ $(a \rightarrow \infty$ $D_f: x > a \text{ or } (a, \infty)$
3	$a \leq x < b$ $A[\text{---})b \rightarrow \text{half-open interval}$ $D_f: a \leq x < b \text{ or } [a, b)$	3	$a \leq x$ $[a \rightarrow \infty$ $D_f: x \geq a \text{ or } [a, \infty)$
4	$a < x \leq b$ $a(\text{---}]b \rightarrow \text{half-open interval}$ $D_f, R_f: a < x \leq b \text{ or } (a, b]$	4	$x < b$ $-\infty \leftarrow b)$ $D_f: x < b \text{ or } (-\infty, b)$
		5	$x \leq b$ $-\infty \leftarrow b]$ $D_f: x \leq b \text{ or } (-\infty, b]$

Example 1: $y = f(x) = x + 15$

$D_f: \mathbb{R}$

To find the range:- Re- write the function as $x=f(y)$

$$y = x + 15 \rightarrow x = f(y) = y - 15$$

$\therefore D_f: \mathbb{R}$

Example 2:

$$y = \frac{3}{x-2} \quad \text{المقام} \neq$$

$$\text{الشرط} \rightarrow x - 2 \neq 0 \rightarrow x \neq 2 \quad \rightarrow \quad \therefore \quad D_f: \mathbf{R}/\{2\}$$

To find the range: - Re- write the function as $x=f(y)$

$$y = \frac{3}{x-2} \rightarrow yx - 2y = 3$$

$$yx = 3 + 2y \rightarrow x = f(y) = \frac{3 + 2y}{y}$$

$$\therefore R_f: \mathbf{R}/\{0\}$$

Example 3:

$$y = \sqrt{x-1} \quad \text{لا يجوز أن تكون القيمة سالبة تحت الجذر (قيمة خيالية)}$$

$$\text{الشرط} \rightarrow x - 1 \geq 0 \rightarrow x \geq 1$$

$$D_f: \mathbf{x: x \geq 1 \text{ or } [1, \infty)}$$

To find the range:- Re- write the function as $x=f(y)$

$$y = \sqrt{x-1} \rightarrow y^2 = x - 1$$

$$x = y^2 + 1$$

نأخذ القيم الموجبة و الصفر فقط لأن الدالة الأصلية هي دالة $R_f: y: y \geq 0 \text{ or } [0, \infty)$ جذرية

Example 4:

$$y = \frac{1}{\sqrt{x-1}} \quad \text{هنا الجذر موجود في مقام الدالة}$$

$$\text{الشرط} \rightarrow x - 1 > 0 \rightarrow x > 1$$

$$D_f: \mathbf{x: x > 1 \text{ or } (1, \infty)}$$

To find the range:- Re- write the function as $x=f(y)$

$$y = \frac{1}{\sqrt{x-1}} \rightarrow y^2 = \frac{1}{x-1}$$

$$y^2x - y^2 = 1 \rightarrow x = \frac{1 - y^2}{y^2} \rightarrow x = 1 + \frac{1}{y^2}$$

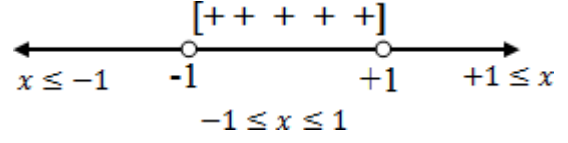
نأخذ القيمة الموجبة فقط لأن الدالة الأصلية هي دالة جذرية و كسرية
 $\therefore R_f: y: y > 0$

Example 5:

$$y = \sqrt{1-x^2}$$

$$\text{الشرط} \rightarrow 1 - x^2 \geq 0 \quad (1)$$

$$-x)(1+x) \geq 0$$



$$\text{من رسم الدالة واختبار الفترات} \rightarrow \therefore D_f: -1 \leq x \leq 1 \text{ or } [-1, 1]$$

To find the range:- Re- write the function as $x=f(y)$

$$y = \sqrt{1-x^2} \rightarrow y^2 = 1 - x^2$$

$$x^2 = 1 - y^2 \rightarrow x = \sqrt{1-y^2}$$

$$\text{الشرط} \rightarrow 1 - y^2 \geq 0 \quad (1)$$

$$-y)(1+y) \geq 0$$

$$\text{من رسم الدالة واختبار الفترات} \rightarrow \therefore R_f: 0 \leq y \leq 1 \text{ or } [0, 1]$$

Example 6:

$$y = x + \frac{1}{x} \quad x \neq 0$$

مجال الدالة الكلي هو حاصل تقاطع مجال الدالتين $D_f: R / \{0\}$

$$\text{To find the Range: } y = x + \frac{1}{x} \rightarrow y = \frac{x^2+1}{x} \rightarrow yx = x^2 + 1$$

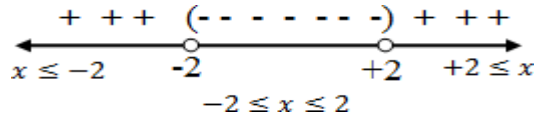
$$x^2 - yx + 1 = 0$$

$$a = 1 \text{ \& } b = -y \text{ \& } c = 1$$

$$x = \frac{y \pm \sqrt{y^2 - (4 \cdot 1 \cdot 1)}}{2 \cdot 1}$$

$$\text{الشرط} \rightarrow y^2 - 4 \geq 0 \quad (y$$

$$-2)(y+2) \geq 0$$



$$\text{من رسم الدالة} \quad R_f: y \geq 2 \cup y \leq -2 \text{ or } R / (-2, 2)$$

Example 7:

$$y = \sqrt{x^2 - 3x}$$

$$\text{الشرط} \rightarrow x^2 - 3x \geq 0$$

$$x(x - 3) \geq 0$$

$$++++ 0 [-----] 3 +++++$$

←

→

من رسم الدالة $\rightarrow D_f: x \geq 3 \cup x \leq 0 \text{ or } R / (0, 3)$

To find the Range:

$$y = \sqrt{x^2 - 3x} \rightarrow y^2 = x^2 - 3x$$

$$x^2 - 3x - y^2 = 0$$

حل المعادلة من الدرجة الثانية بطريقة الدستور $= 1 \& b = -3 \& c = -y^2$

$$x = \frac{3 \pm \sqrt{9 - [4 \cdot 1 \cdot (-y^2)]}}{2 \cdot 1} \rightarrow 9 + 3y^2 \geq 0$$

المقابل $R_f: y \geq 0 \text{ or } [0, \infty)$

اصل الدالة عي دالة جذرية ولذا يتم استثناء القيم السالبة من المجال

Example 8:

Find the domain only for $y = \sqrt{x - 3} + \sqrt{3 - 2x}$

$$\sqrt{x - 3} + \sqrt{3 - 2x}$$

↓

↓

$$Df1 \Rightarrow x - 3 \geq 0 \rightarrow x \geq 3$$

$$Df2 \Rightarrow 3 - 2x \geq 0 \Rightarrow 3 \geq 2x \rightarrow \frac{3}{2} \geq x$$

$$+++3/2]----- [3+++$$

← →

$\therefore D_f = \emptyset$ من رسم مجال الدوال واختبار الفترات

كمية خالية

$$(Df1) \cap (Df2) = \emptyset$$

Example 9: Find the domain only for

$$y = \sqrt{4 - x} + \sqrt{x - 1}$$

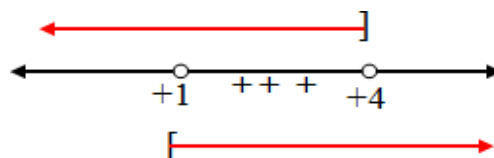
$$= \sqrt{4 - x} + \sqrt{x - 1}$$

↓

↓

$$4 - x \geq 0$$

$$x - 1 \geq 0$$



$$D_f :- 1 \leq x \leq 4 \text{ or } [1, 4]$$

$$4 \geq x \quad x \geq 1$$

من رسم مجال الدوال واختبار الفترات $\therefore D_f :- 1 \leq x \leq 4$ or $[1,4]$

Example 10:

Find the domain only for:-

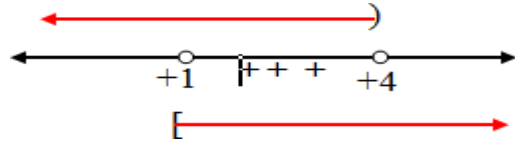
$$y = \sqrt{\frac{x-1}{4-x}}$$

$$y = \frac{\sqrt{x-1}}{\sqrt{4-x}}$$

$$Df1 \Rightarrow x - 1 \geq 0 \Rightarrow x \geq 1$$

$$Df2 \Rightarrow 4 - x > 0 \Rightarrow 4 > x$$

$$\therefore D_f :- 1 \leq x < 4$$
 or $[1,4)$



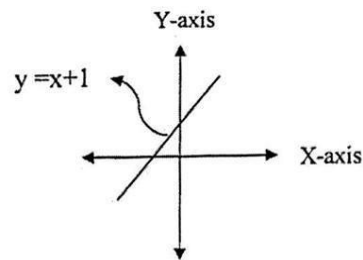
Note: the domain and the range of any function could be found also by graphing the function itself.

The projection of the graph of a function (f) on the x-axis is the domain (D_f) and on the y-axis is the range (R_f).

Example 11: $y = x + 1$

$$D_f: \mathbb{R}$$

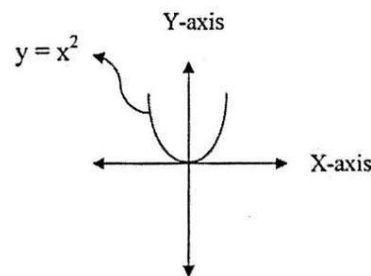
$$R_f: \mathbb{R}$$



Example 12: $y = x^2$

$$D_f: \mathbb{R}$$

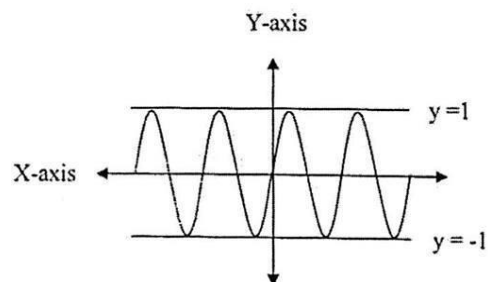
$$R_f: \mathbb{R}$$



Example 13: $y = \sin x$

$$D_F: \mathbb{R}$$

$$R_F = [-1, 1]$$

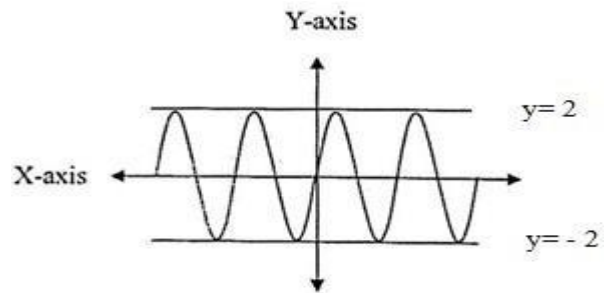


Example 14:

$$y = 2\sin x$$

$$D_f: \mathbb{R}$$

$$R_f = [-2, 2]$$

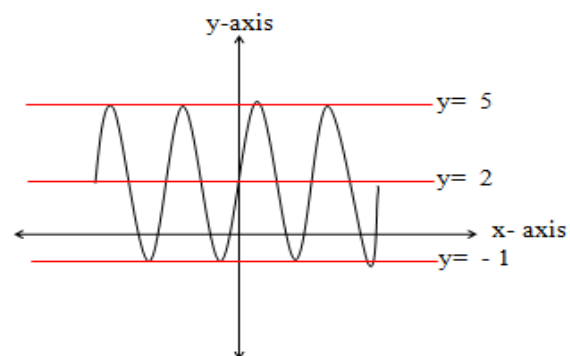


Example 15:

$$y = 2 + 3\sin x$$

$$D_f: \mathbb{R}$$

$$R_f = [-1, 5]$$

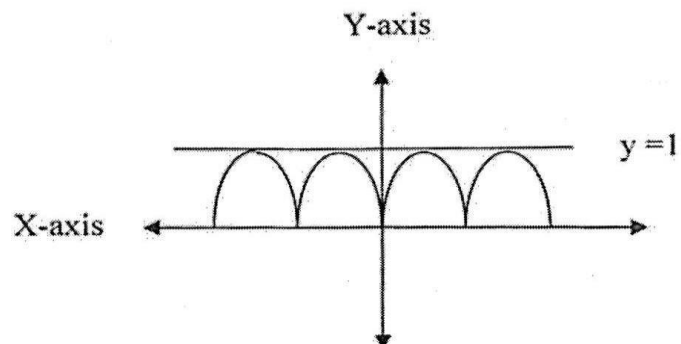


Example 16:

$$y = \sin^2 x$$

$$D_f: \mathbb{R}$$

$$R_f = [0, 1]$$

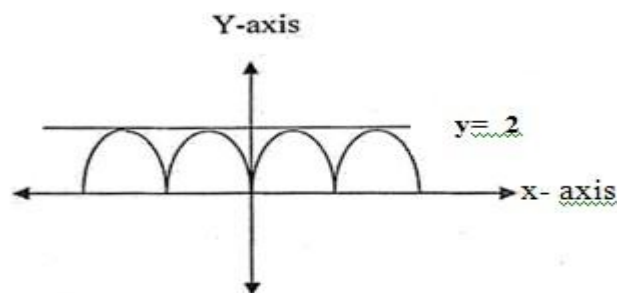


Example 17:

$$y = -2\sin^2 x$$

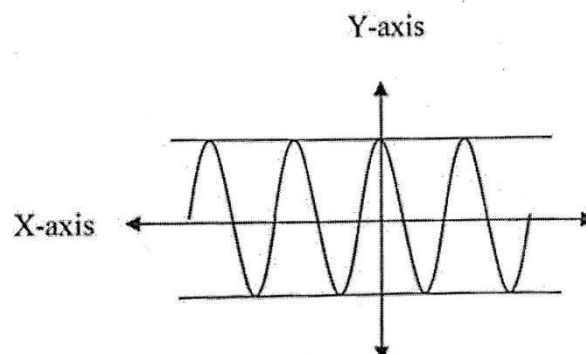
$$D_f: \mathbb{R}$$

$$R_f = [-2, 0]$$



Example 18:

$$y = \cos x$$



$$D_f: \mathbb{R}$$

$$R_f = [-1, 1]$$

1-2-2 Even and odd Functions: الدوال الزوجية و الدوال الفردية

A function (f) is called:

- Even if $\rightarrow f(-x) = +f(x)$
- Odd if $\rightarrow f(-x) = -f(x)$

Examples:

1) $y = x^2$

$$f(-x) = (-x)^2 = x^2 = +f(x) \rightarrow \therefore \text{even function}$$

2) $y = x^3$

$$f(-x) = (-x)^3 = -x^3 = -f(x) \rightarrow \therefore \text{odd function}$$

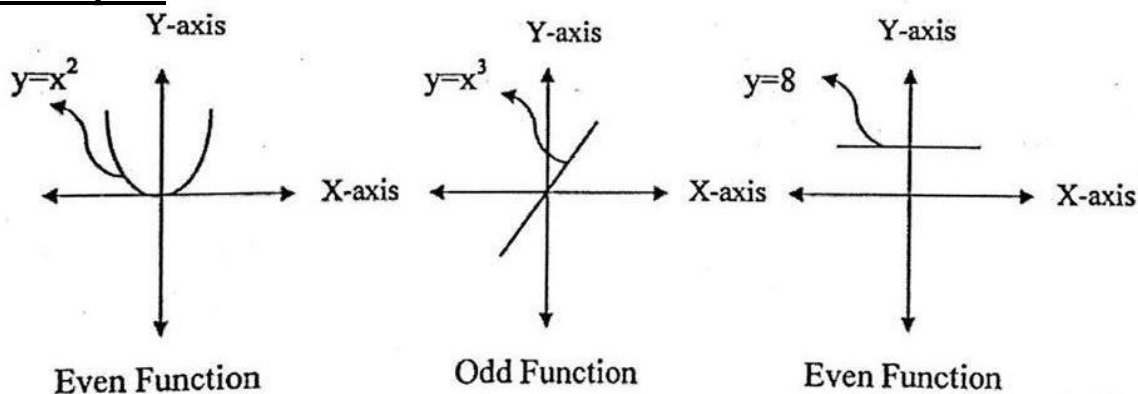
3) $y = 8$

$$f(-x) = 8 \rightarrow \therefore \text{even function}$$

Note (1) : For all $x \in D_f$:

- Even if $\rightarrow$ the function is symmetric about the y-axis.
- Odd if $\rightarrow$ the function is symmetric about the origin.

Examples:



Note (2) :

- $Odd \pm Odd = Odd$ & $Even \pm Even = Even$
- $Odd * Odd = Even$ & $Even * Even = Even$
- $Odd/Even = Odd$ & $Even/Odd = Odd$
- $Odd * Even = Odd$ & $Even * Odd = Even$

Examples

1) $y = \sin x$

Odd function (symmetric about the origin)

2) $y = \cos x$

Even function (symmetric about the y-axis)

3) $y = \tan x$

$$y = \frac{\sin x}{\cos x} = \frac{\text{Odd}}{\text{Even}} = \text{Odd function}$$

4) $f(x) = \frac{x^2 + x^4}{x + \sin x}$

$$f(x) = \frac{\text{even} + \text{Even}}{\text{Odd} + \text{Odd}} = \frac{\text{Even}}{\text{Odd}} = \text{Odd function}$$

5) $f(x) = x^3 - 2$

$f(x) = \text{Odd} - \text{Even} = \text{neither Even nor Odd}$

6) $f(x) = \frac{x^3 + x^5}{\sin x + 2}$

$$f(x) = \frac{\text{Odd} + \text{Odd}}{\text{Odd} + \text{Even}} = \text{neither Even nor Odd}$$

1-2-3 Limits of a function: النهايات

Definition

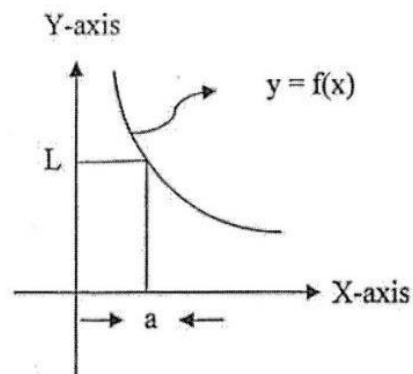
$\lim_{x \rightarrow a} f(x) = L$ Mean that when a value of (x)

Close to (a) $\Rightarrow f(x)$ approaches the limiting value (L).

$\lim_{x \rightarrow a+} f(x) = L$ Mean that (x) approaches (a) from the

$\lim_{x \rightarrow a-} f(x) = L$ Mean that (x) approaches (a) from the

Note:



If $\lim_{x \rightarrow a+} f(x) = L = \lim_{x \rightarrow a-} f(x) = L$ we say that $\lim_{x \rightarrow a} f(x) = L$ exist,

Otherwise the limit doesn't exist.

Example 1:

Find $\lim_{x \rightarrow 1} f(x)$ when $f(x) = \begin{cases} x^2 + 1 & \text{when } x \geq 1 \\ 3 & \text{when } x < 1 \end{cases}$

$$\lim_{x \rightarrow 1-} f(x) = \lim_{x \rightarrow 1-} (x^2 + 1) = (1)^2 + 1 = 2$$

$$\lim_{x \rightarrow 1+} f(x) = \lim_{x \rightarrow 1+} (3 - x) = 3 - 1 = 2$$

$$\therefore \lim_{x \rightarrow 1-} f(x) = \lim_{x \rightarrow 1+} f(x) = 2 \rightarrow \therefore \lim_{x \rightarrow 1} f(x) \text{ Exist}$$

Example 2:

Find $\lim_{x \rightarrow 0} f(x)$ where $f(x) = \begin{cases} x^2 + 1 & \text{when } x \geq 0 \\ x & \text{when } x < 0 \end{cases}$

$$\lim_{x \rightarrow 0+} f(x) = \lim_{x \rightarrow 0+} (x^2 + 1) = 0 + 1 = 1$$

$$\lim_{x \rightarrow 0-} f(x) = \lim_{x \rightarrow 0-} (x) = 0$$

$$\therefore \lim_{x \rightarrow 0+} f(x) \neq \lim_{x \rightarrow 0-} f(x) \rightarrow \therefore \lim_{x \rightarrow 0} f(x) \text{ Doesn't exist}$$

Properties of Limits: خصائص النهايات

Let: $\lim_{x \rightarrow a} f(x) = L1$

$\lim_{x \rightarrow a} g(x) = L2$

K is a constant, then:

$$1) \lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = L1 \pm L2$$

$$2) \lim_{x \rightarrow a} [f(x) * g(x)] = \lim_{x \rightarrow a} f(x) * \lim_{x \rightarrow a} g(x)$$

$$3) \lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \text{ but } \lim_{x \rightarrow a} g(x) \neq 0$$

$$4) \lim_{x \rightarrow a} K * f(x) = K * \lim_{x \rightarrow a} f(x)$$

$$5) \lim_{x \rightarrow a} K = K$$

- 6) $\lim_{x \rightarrow a} x = a$
- 7) $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ and $\lim_{x \rightarrow \infty} \sqrt{x} = \infty$
- 8) $\lim_{x \rightarrow 0+} \frac{1}{x} = +\infty$ and $\lim_{x \rightarrow 0-} \frac{1}{x} = -\infty$ but $\lim_{x \rightarrow 0} \frac{x}{1} = 0$
- 9) $\lim_{x \rightarrow 0} \sin x = 0$; $\lim_{x \rightarrow 0} \cos x = 1$ and $\lim_{x \rightarrow 0} \tan x = 0$
- 10) $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ and $\lim_{x \rightarrow 0} \frac{x}{\sin x} = 1$
- 11) $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$ and $\lim_{x \rightarrow 0} \frac{x}{\tan x} = 1$
- 12) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$
- 13) $\lim_{x \rightarrow a} \sin \left(\frac{x^2}{\pi + x} \right) = \sin \left(\lim_{x \rightarrow a} \frac{x^2}{\pi + x} \right)$

Note: sin or cos or any trigonometric function is the same

14) $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$ and $\lim_{x \rightarrow a} \frac{1}{x^n} = \left[\lim_{x \rightarrow a} \frac{1}{x} \right]^n$

Note:

Undefined expression in limits:

$$\frac{0}{0}, \frac{\infty}{\infty}, \frac{0}{\infty}, \frac{\infty}{0}, 0 * \infty, \infty * \infty, \infty - \infty$$

but we can say $\infty + \infty = \infty$

أساليب الحل الممكن اتباعها في حل أسئلة الغايات:

- 1) التعويض المباشر اذا كان الناتج معرف.
- 2) باستخدام طرق التحليل المختلفة أو الضرب بالمرافق اذا كانت ليست دوال مثلثية. 3) باستخدام الخصائص من 9 الى 13 اذا كانت دوال مثلثية.
- 4) اذا كانت $x \rightarrow \infty$ فهناك ثلاث طرق للحل:
أولاً: اذا كانت دوال مثلثية نحول x الى متغير آخر و ليكن 1 مثلاً و عندما $x \rightarrow \infty$ فإن $y \rightarrow 0$
ثانياً: اذا كانت دوال كسرية نقسم على اكبر اس موجود في المقام.
ثالثاً: اذا كانت دوال غير كسرية و ليست مثلثية نحولها الى دوال كسرية بالضرب في المرافق ثم نقسم على اكبر اس موجود في المقام.
- 5) باستخدام طريقتين أو أكثر.

Evaluate the following limits:

Examples:

$$1) \lim_{x \rightarrow 1} \frac{x^2+1}{x^2-1} = \frac{1+1}{1} = 2$$

$$2) \lim_{x \rightarrow 1} \frac{x^2-1}{x+1}$$

$$\lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{(x+1)} = \lim_{x \rightarrow 1} (x-1) = -1 - 1 = -2$$

$$3) \lim_{x \rightarrow 1} \frac{x^3-1}{x-1}$$

$$= \lim_{x \rightarrow 1} \frac{(x-1)(x^2+x+1)}{(x-1)} = \lim_{x \rightarrow 1} (x^2+x+1) = 1+1+1 = 3$$

$$4) \lim_{x \rightarrow 0} \frac{x}{1-\sqrt{1-x}}$$

$$= \lim_{x \rightarrow 0} \frac{x}{1-\sqrt{1-x}} * \frac{1+\sqrt{1-x}}{1+\sqrt{1-x}} \quad \text{الضرب في مرافق المقام}$$

$$= \lim_{x \rightarrow 0} \frac{x(1+\sqrt{1-x})}{1-(1-x)} = \lim_{x \rightarrow 0} \frac{x(1+\sqrt{1-x})}{1-1+x} = \lim_{x \rightarrow 0} (1+\sqrt{1-x}) = 1+\sqrt{1} = 2$$

$$5) \lim_{x \rightarrow 0} \frac{\sin 3x}{x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin 3x}{x} * \frac{3}{3} = 1 * 3 = 3$$

$$6) \lim_{x \rightarrow 0} \frac{\sin^2 3x}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 3x}{x^2} * \frac{3*3}{3*3} = 9 * 1 * 1 = 9$$

$$7) \lim_{x \rightarrow 1} \frac{\sin 3x}{\sin 5x}$$

$$= \lim_{x \rightarrow 1} \frac{\sin 3x * \frac{3x}{5x}}{\sin 5x * \frac{5x}{5x}} = \frac{3}{5}$$

$$8) \lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1+\cos x)} = \lim_{x \rightarrow 0} \frac{\sin x}{x} * \lim_{x \rightarrow 0} \frac{\sin x}{(1+\cos x)} = 1 * \frac{0}{2} = 0$$

$$\begin{aligned}
 9) \quad & \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \\
 &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} * \frac{1 + \cos x}{1 + \cos x} = \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{\sin x}{x} * \lim_{x \rightarrow 0} \frac{\sin x}{(1 + \cos x)} = 1 * \frac{0}{2} = 0
 \end{aligned}$$

$$10) \quad \lim_{x \rightarrow \pi} \frac{\sin x}{x - \pi}$$

Assume $y = x - \pi \Rightarrow x = y + \pi$

when $x \rightarrow \pi \Rightarrow y \rightarrow 0$

$$= \lim_{y \rightarrow 0} \frac{\sin(y + \pi)}{y} = \lim_{y \rightarrow 0} \frac{\sin y \cos \pi + \cos y \sin \pi}{y} = \lim_{y \rightarrow 0} \frac{-\sin y}{y} = -1$$

$$\begin{aligned}
 11) \quad & \lim_{x \rightarrow 2} \frac{\cos \frac{\pi}{x}}{x - 2} \\
 &= \lim_{x \rightarrow 2} \frac{\sin(\frac{\pi}{2} - \frac{\pi}{x})}{x - 2}
 \end{aligned}$$

Assume $y = x - 2 \Rightarrow x = y + 2$

when $x \rightarrow 2 \Rightarrow y \rightarrow 0$

$$\begin{aligned}
 &= \lim_{y \rightarrow 0} \frac{\sin(\frac{\pi}{2} - \frac{\pi}{y+2})}{y} = \lim_{y \rightarrow 0} \frac{\sin(\frac{y\pi + 2\pi - 2\pi}{2(y+2)})}{y} \\
 &= \lim_{y \rightarrow 0} \frac{\sin(\frac{y\pi}{2(y+2)})}{y} * \frac{\frac{\pi}{2(y+2)}}{\frac{\pi}{2(y+2)}} = \lim_{y \rightarrow 0} \frac{\pi}{2(y+2)} = \frac{\pi}{4}
 \end{aligned}$$

Example 11: $\lim_{x \rightarrow \infty} x \sin \left(\frac{2}{x} \right)$

Assume $x = \frac{1}{y} \rightarrow y = \frac{1}{x}$

when $x \rightarrow \infty \Rightarrow y \rightarrow 0$

$$= \lim_{y \rightarrow 0} \frac{1}{y} \sin 2y$$

$$= \lim_{y \rightarrow 0} \frac{\sin 2y}{y} * \frac{2}{2} = 2$$

(الاقتراب من الـ ∞)

Example 12: $\lim_{x \rightarrow \infty} \frac{4x^3 - 2x^2 + 1}{3x^3 - 5}$

$$= \lim_{x \rightarrow \infty} \frac{\frac{4x^3}{x^3} - \frac{2x^2}{x^3} + \frac{1}{x^3}}{\frac{3x^3}{x^3} - \frac{5}{x^3}} = \frac{4}{3}$$

ملاحظة: في حال كون الدالة كسرية والاقتراب من الـ ∞ ، نقسم البسط والمقام على اكبر اس في المقام

Example 14: $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 2x} - x)$

$$\begin{aligned} &= \lim_{x \rightarrow \infty} (\sqrt{x^2 + 2x} - x) * \frac{\sqrt{x^2 + 2x} + x}{\sqrt{x^2 + 2x} + x} = \lim_{x \rightarrow \infty} \frac{x^2 + 2x - x^2}{\sqrt{x^2 + 2x} + x} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{2x}{x}}{\sqrt{\frac{x^2}{x^2} + \frac{2x}{x^2}} + \frac{x}{x}} = \frac{2}{2} = 1 \end{aligned}$$

Continuity of a function:

Continuity of a moving particle on a single path without unbroken curve, gaps, jumps, or holes such curve can be said to be as continuous.

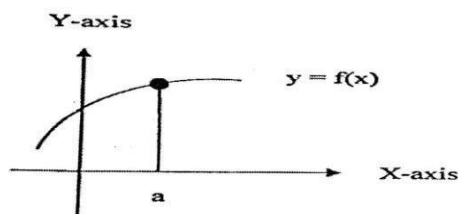


Figure (1) continuous function

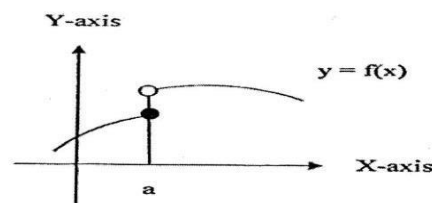


Figure (2) not continuous function

A function is said to be continuous at $x = a$ if the following are satisfied:

- 1) The $f(x)$ is exist or defined.
- 2) $\lim_{x \rightarrow a} f(x)$ exist.
- 3) $\lim_{x \rightarrow a} f(x) = f(a)$

Otherwise the function is not continuous.

Example 1: check if the function is continuous at $x = 5$, & $x = 0$

$$\text{where } f(x) = \begin{cases} x^2 - 1 & \text{When } x \geq 5 \\ x & \text{When } x < 5 \end{cases}$$

at $x = 5$

$$f(5) = (5)^2 - 1 = 24$$

$$\lim_{x \rightarrow 5+} x^2 - 1 \rightarrow \lim_{x \rightarrow 5+} 25 - 1 = 24$$

$$\lim_{x \rightarrow 5-} \rightarrow \lim_{x \rightarrow 5-} 5 = 5$$

$$\lim_{x \rightarrow 5+} f(x) \neq \lim_{x \rightarrow 5-} f(x)$$

The limit does not exist, therefore the function not continuous at $x = 5$

At $x = 0$

$f(0)$ = does not exist, therefore the function not continuous at $x = 0$

Example 2: Find the constant (a) and (b) if the function is:

$$f(x) = \begin{cases} x^2 + a & \text{When } x \geq 0 \\ 3 + b & \text{When } -1 \leq x < 0 \\ x + 5 & \text{When } x < -1 \end{cases}$$

And the function is continuous at $x = 0$ and $x = -1$

$$f(0) = (0)^2 - a = a$$

$$\lim_{x \rightarrow 0+} x^2 + a = a$$

$$\lim_{x \rightarrow 0-} 3 + b = 3 + b$$

$$\lim_{x \rightarrow -1-} x + 5 = 4$$

The limit must be exist so $4 = 3 + b$ then $b = 1$

Then the function equal the limit value $4 = 3 + b$ then $b = 1$ substitute in equation (1) then $a = 3 + 1 = 4$

Example : For $x \neq 2$ the function is equal to $\frac{x^2+3x-10}{x-2}$, find the value of $f(2)$ to make the function continuous at $x = 2$.

The limit to be exist must be equal from the left to the right so:

$$\lim_{x \rightarrow 2} \frac{x^2+3x-10}{x-2} \rightarrow \lim_{x \rightarrow 2} \frac{(x+5)(x-2)}{(x-2)}$$

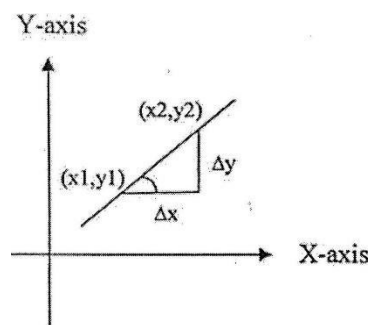
$$\lim_{x \rightarrow 2} (x + 5) = 7 \quad \text{To be continuous then:} \quad f(x) = \lim_{x \rightarrow 2} f(x) = 7$$

1-3 Equation of straight lines and circles:

A. Equation of Straight lines:

- The slope (m) of a straight line passing through points (x_1, y_1) & (x_2, y_2) is:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{y_1 - y_2}{x_1 - x_2}$$



- The equation of a straight line passing through (x_1, y_1) and has a slope (m) is:

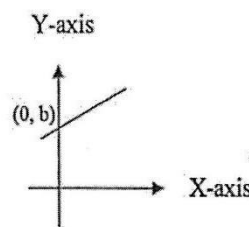
$$y - y_1 = m(x - x_1) \dots\dots\dots (\text{the point - slope equation of the line})$$

- The general formula for the equation of a straight line with a slope (m)

$$y - y_1 = m(x - x_1)$$

$$y - b = m(x - 0)$$

$$y = mx + b \dots\dots\dots (\text{the point - intercept equation of the line})$$



Note 1: Two lines are parallel if and only if they have the same slopes.

$$L1 \text{ parallel } L2 \text{ if } m_1 = m_2$$

Note 2: Two lines are perpendicular if and only if they have the product of their slopes is (-1).

$$L1 \text{ perpendicular } L2 \text{ if } m_1 \times m_2 = -1 \rightarrow m_1 = \frac{-1}{m_2}$$

Example 1: Find the equation of the line passing through $(-2, -1)$ & $(3, 4)$?\

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-1)}{3 - (-2)} = 1$$

Using $(-2, -1)$ we find:

$$y - y_1 = m(x - x_1)$$

$$y + 1 = 1(x + 2)$$

$$y = x + 1$$

$$\text{Or using } (3, 4) \text{ we find: } y - 4 = 1(x - 3) \therefore y = x + 1$$

Example 2: Find the slope and y-intercept for $8x + 5y = 20$

$$8x + 5y = 20$$

$$5y = -8x + 20$$

$$y = \frac{-8}{5}x + 4$$

$$m = \frac{-8}{5} \text{ \& } b = 4$$

Example 3: Find the equation of the line passing through the origin and the point of intersection of $\rightarrow x + y = 2$ & $L2 \rightarrow 2x - y = -5$?

$$x + y = 2 \rightarrow x = 2 - y$$

$$2(2 - y) - y = -5$$

$$y = 3 \rightarrow x = -1$$

The point of intersection $(-1, 3)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 0}{-1 - 0} = -3$$

Using $(0, 0)$ or $(-1, 3)$ we find:

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -3(x - 0) \rightarrow y + 3x = 0$$

B. Equation of circle:

The equation of a circle centered at (h, k) and has a radius (r) is:

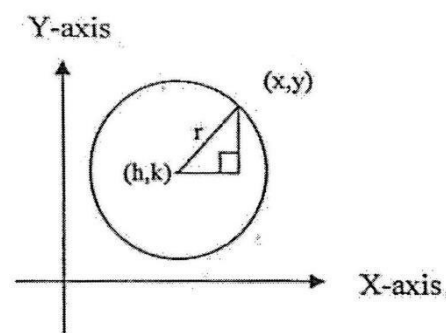
$$(x - h)^2 + (y - k)^2 = r^2$$

Example 1: Find the radius and coordinate of center for:

$$x^2 + y^2 + 4x - 2x - 2y + 1 = 0$$

$$x^2 + 4x + 4 = 4 + y^2 - 2y + 1 = 0$$

$$(x + 2)(x + 2) - 4 + (y - 1)(y - 1) = 0$$



$$(x + 2)^2 + (y - 1)^2 = 4$$

The coordinate of center is $(-2, 1)$ and the radius is (2) unit length **Or**

$$x^2 + y^2 + 4x - 2y + 1 = 0 \rightarrow eq: x^2 + y^2 + ax + by + c = 0$$

$$h = \frac{-(-4)}{2} = -2 \rightarrow h = \frac{-(a)}{2}$$

$$k = \frac{-(-2)}{2} = 1 \rightarrow k = \frac{-(b)}{2}$$

$$r = \sqrt{(-2)^2 + (1)^2 - 1} = 2 \rightarrow r = \sqrt{h^2 + k^2 - c}$$

Example 2: Find the equation of the circle centered at $(1, -2)$ and passing through the point $(7, 4)$?

$$d = r = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{(7 - 1)^2 + (4 + 2)^2}$$

$$= \sqrt{72} \text{ unit length}$$

The equation is:

$$(x - 1)^2 + (y + 2)^2 = 72$$

Example 3: For this equation: $3y^2 - 12y + 3x^2 + 6x = 18$ find:

- The center and the radius of the circle.
- The equation of the circle.
- The area of the circle.

$$3y^2 - 12y + 3x^2 + 6x = 18$$

$$[3x^2 + 3y^2 + 6x - 12y - 18 = 0] \div 3$$

$$x^2 + y^2 + 2x - 4y - 6 = 0$$

$$h = \frac{-(a)}{2} = \frac{-(-2)}{2} = -1 \text{ \& } k = \frac{-(b)}{2} = \frac{-(-4)}{2} = 2$$

The point of the circle is:

$$(x + 1)^2 + (y - 2)^2 = 11$$

The area of circle is: $A = \pi r^2 = 3.14 * 11 = 34.54$ unit area

Ex: Find the slope (m) and y-intercept (b)

$$1) 8x + 5y = 20$$

$$5y = -8x + 20$$

$$y = -\frac{8}{5}x + \frac{20}{5}$$

$$y = mx + b$$

$$\therefore m = -\frac{8}{5}$$

$$2) x - 2y = 4$$

$$2y = x - 4$$

$$y = \frac{1}{2}x - \frac{4}{2}$$

$$m = \frac{1}{2}$$

$$b = -\frac{4}{2} = -2$$

$$b = \frac{20}{5} = 4$$

Ex: Find the equation of the line passing through the origin and the point of intersection of $L_1: x + y = 2$ and $L_2: 2x - y = -5$

$$x + y = 2$$

$$2x - y = -5$$

$$3x = -3 \rightarrow x = -1 \rightarrow y = 3 \rightarrow (-1, 3)$$

$$m = \frac{\Delta y}{\Delta x} = \frac{3-0}{-1-0} = -3$$

$$y - y_0 = m(x - x_0) \rightarrow y - 0 = -3(x - 0)$$

Note:

❖ Two lines are parallel if and only if they have the same slope $L_1 // L_2$ if

$$m_1 = m_2$$

❖ Two lines are perpendicular (orthogonal, vertical) if and only if the product of their slopes is (-1)

$$L_1 \perp L_2 \text{ if } m_1 * m_2 = -1 \text{ or } m_1 = -\frac{1}{m_2}$$

A) Circle: the equation of the circle with a center (h, k) and has a radius (r)

$$\text{is } (x - h)^2 + (y - k)^2 = r^2$$

$$(x - 2)^2 + (y - k)^2 = 9$$

اوجد المركز و نصف القطر

$$3 = \text{نق}, (2, -3) \text{ المركز}$$

$$(x + 2)^2 + (y + 3)^2 = 16 \quad 4 = \text{نق}, (-2, -3) \text{ المركز}$$

Ex: Find the radius and coordinate of the center

$$x^2 + y^2 + 4x - 2y + 1 = 0$$

$$x^2 + 4x + y^2 - 2y = -1$$

$$x^2 + 4x + 4 - 4 + y^2 - 2y + 1 - 1 = -1 \quad \text{نضيف و نطرح مربع } x \text{ معامل } \frac{1}{2}$$

$$(x + 2)^2 - 4 + (y - 1)^2 - 1 = -1 \quad \text{نق المركز}$$

$$(x + 2)^2 + (y - 1)^2 = 4 \rightarrow (-2, 1), 2$$

Ex: For what value of k does the circle $(x - k)^2 + (y - 2k)^2 = 10$ pass through the point (1, 1)

$$(1 - k)^2 + (1 - 2k)^2 = 10$$

$$1 - 2k + k^2 + 1 - 4k + 4k^2 - 10 = 0$$

$$5k^2 - 6k - 8 = 0$$

$$(5k + 4)(k - 2) = 0 \quad \text{either } k = -\frac{4}{5} \text{ or } k = 2$$

Ex: Find the equation of the circle centered at (1,-2) and passing through (7,4)

$$r = \sqrt{(7-1)^2 + (4-(-2))^2} = \sqrt{72}$$

The equation is $(x-1)^2 + (y+2)^2 = 72$

H.W:

- ❖ Find the equation of the circle that passes through the points A(2, 3), B(-4, 3) and C(3, 2).
- ❖ Find the equation of the circle which passes through (10, 2), (9, -3) and the center of the circle lies on the y-axis?

هذا يعني ان احداثيات المركز $(0, k)$

(2) Differentiation

We call that derivate of the function $f(x)$ as $f'(x)$, $\frac{df(x)}{dx}$ or $\frac{dy}{dx}$ or y'

- ❖ The derivative of a function at a point $x = a$ is the slope of the tangent line to the curve.

عند نقطة التماس ميل المماس = مشتقة الدالة

Ex: Find the equation of the tangent line to the curve

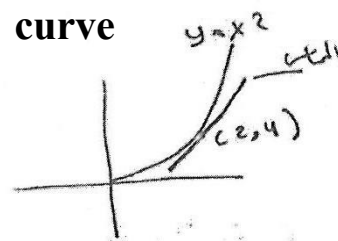
$$y = x^2 \text{ at } (2, 4)$$

$$\frac{dy}{dx} = 2x$$

$$m = \frac{dy}{dx} = 2 * 2 = 4 \text{ at } (2, 4)$$

تحقق معادلة المماس (2,4)

$$y - y_0 = m(x - x_0) \quad \therefore y - 4 = 4(x - 2) \quad \text{معادلة المماس عند } (2, 4)$$



2-1 Properties of the derivative

- 1) $\frac{d}{dx}(c) = 0$ where c is a constant
- 2) $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
- 3) $\frac{d}{dx} c f(x) = c f'(x)$
- 4) $\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x)$
- 5) $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$
- 6) $\frac{d}{dx}(x^u) = ux^{u-1}$

$$7) \frac{d}{dx} (\sin x) = \cos x$$

$$8) \frac{d}{dx} (\cos x) = -\sin x$$

$$9) \frac{d}{dx} (\tan x) = \sec^2 x$$

$$10) \frac{d}{dx} (\cot x) = -\csc^2 x$$

$$11) \frac{d}{dx} (\sec x) = \sec x \tan x$$

$$12) \frac{d}{dx} (\csc x) = -\csc x \cot x$$

Example: Find $\frac{dy}{dx}$ for the following

$$\diamond y = x^2(x^3 + 2) \Rightarrow y' = x^2(3x^2) + (x^3 + 2) * 2x$$

$$\diamond y = \frac{3}{x} \Rightarrow y' = \frac{x*0 - 3*1}{x^2} = -\frac{3}{x^2}$$

$$\text{OR } y = 3x^{-1} \Rightarrow y' = -3x^{-2} = -\frac{3}{x^2}$$

$$\diamond y = \tan^2 x \Rightarrow y' = 2 \tan x \cdot \sec^2 x$$

$$\diamond y = \sin^2 x^3 = (\sin x^3)^2$$

$$y' = 2 \sin x^3 \cos x^3 * 3x^2$$

$$\diamond y = \sqrt{\sec(3x^3)} \Rightarrow y = (\sec(3x^3))^{1/2}$$

$$y' = \frac{1}{2} [\sec(3x^3)]^{-1/2} * \sec 3x^3 \tan 3x^3 * 9x^2$$

$$\diamond y = \sqrt{\sec(x \cos x)} \Rightarrow y = [\sec(x \cos x)]^{1/2}$$

$$y' = \frac{1}{2} [\sec(x \cos x)]^{-1/2} * \sec(x \cos x) \tan(x \cos x) *$$

$$[x(-\sin x) + \cos x (1)]$$

$$\diamond y = \tan^2 \left(\cos \frac{1}{x} \right) = \left[\tan \left(\cos \frac{1}{x} \right) \right]^2$$

$$y' = 2 \tan \left(\cos \frac{1}{x} \right) \cdot \sec^2 \left(\cos \frac{1}{x} \right) * -\sin \frac{1}{x} * -x^{-2}$$

2-2 Higher order derivative

Find $\frac{d^4 y}{dx^4}$ for $y = x^6 - 3x^4 + \cos x$

$$y' = 6x^5 - 12x^3 - \sin x$$

$$y'' = 30x^4 - 36x^2 - \cos x$$

$$y''' = 120x^3 - 72x + \sin x$$

$$y'''' = 360x^2 - 72 + \cos x$$

$$\frac{d^2 y}{dx^2} = y'' = \frac{d}{dx} y' = \frac{d}{dx} \frac{dy}{dx}$$

$$\frac{d^3 y}{dx^3} = y''' = \frac{d}{dx} \frac{d^2 y}{dx^2}$$

Chain rule:

If $y = f(t)$ and $x = g(t)$ then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

If $y = f(t)$ and $t = g(x)$ then $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

Ex: if $y = \sin t$, $x = \cos t$ find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{-\sin t} = -\cot t$$

Ex: Find $\frac{dy}{dx}$ if $y = t^3$ and $t = x^2 + 2$

$$\frac{dy}{dt} = 3t^2, \frac{dt}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 3t^2 * 2x = 3(x^2 + 2)^2 * 2x \quad \text{Or}$$

$$= (x^2 + 2)^3 \quad \text{بالتعويض المباشر}$$

$$\frac{dy}{dx} = 3(x^2 + 2)^2 * 2x$$

Ex: Find $\frac{d^2 y}{dx^2}$ if $y = (t^2 + 1)^4$, $x = t^2 + 5$

$$\frac{dy}{dx} = 4(t^2 + 1)^3 * 2t, \frac{dx}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4(t^2+1)^3 \cdot 2t}{2t} = 4(t^2 + 1)^3$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{12(t^2+1)^2 \cdot 2t}{2t} = 12(t^2 + 1)^2$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

2-3 Implicit differentiation الاشتقاق الضمني

We differentiate both sides with respect to x

Ex: Find $\frac{dy}{dx}$ if $x^2 + xy + y - x = 0$

$$2x + xy' + y \cdot 1 + y' - 1 = 0$$

$$2x + y - 1 = -xy' - y'$$

$$2x + y - 1 = y'(-x - 1)$$

$$y' = \frac{2x+y-1}{-x-1}$$

Ex: $\sin y + x \sin x = 1$

$$\cos y \cdot y' + x \cos x + \sin x = 0$$

$$y' \cos y = -x \cos x - \sin x$$

$$y' = \frac{-x \cos x - \sin x}{\cos y}$$

Ex: If $x^2 + y^2 = 1$ show that $\frac{d^2y}{dx^2} = -\frac{1}{y^3}$

$$2x + 2y \cdot \frac{dy}{dx} = 0 \Rightarrow 2y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{y - x \frac{dy}{dx}}{y^2} = -\frac{y - x(-\frac{x}{y})}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{y+\frac{x^2}{y}}{y^2} = -\frac{y^2+x^2}{y^3}$$

$$\frac{d^2y}{dx^2} = -\frac{y^2+x^2}{y^3} = -\frac{1}{y^3}$$

Ex: Find the equation of the tangent to the curve

$$y = \sin \sqrt{x} \text{ at } (\pi^2, 0)$$

$$\frac{dy}{dx} = \cos \sqrt{x} * \frac{1}{2\sqrt{x}} = \cos \pi * \frac{1}{2\pi} = -\frac{1}{2\pi} \text{ at } (\pi^2, 0)$$

$$y - y_0 = m(x - x_0) \Rightarrow y - 0 = -\frac{1}{2\pi}(x - \pi^2)$$

2-4 Application of Derivatives:-

Increasing function when $f'(x) > 0$ المشتقة موجبة

Decreasing function when $f'(x) < 0$ المشتقة سالبة

Horizontal tangent عندما $f'(x) = 0$

و هذا يعني ان مماس الدالة افقي و ان الدالة تعتبر من متزايدة الى متناقصة او بالعكس.

Ex: Graph the function $y = f(x) = x^3 - 3x^2 + 4$

1) مجد نقاط التقاطع مع المحور

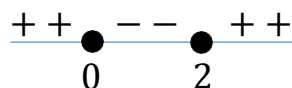
$$\text{When } x = 0 \Rightarrow y = 4 \Rightarrow (0, 4)$$

$$\text{When } y = 0 \Rightarrow 0 = (x + 1)(x - 2)^2 \Rightarrow x = -1 \Rightarrow (-1, 0)$$

$$x = 2 \Rightarrow (2, 0)$$

2) نجد الفترات التي تكون فيها المشتقة الاولى موجبة، سالبة، صفر

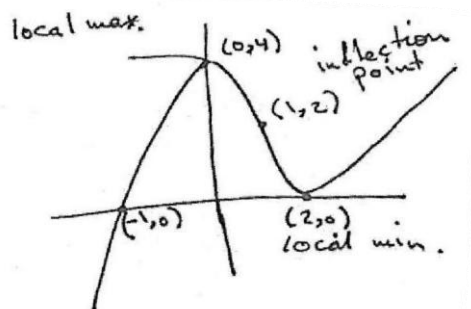
$$y' = 3x^2 - 6x = 3x(x - 2)$$



3) نجد المشتقة الثانية

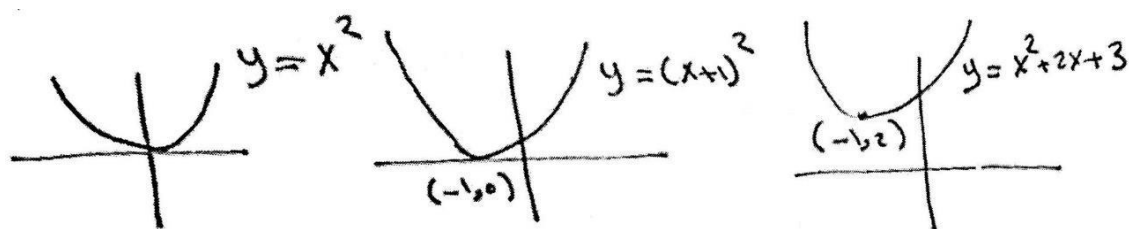
$$y'' = 6x - 6 \Rightarrow x = 1$$

x	y
-1	0
0	4
1	2
2	0
3	4

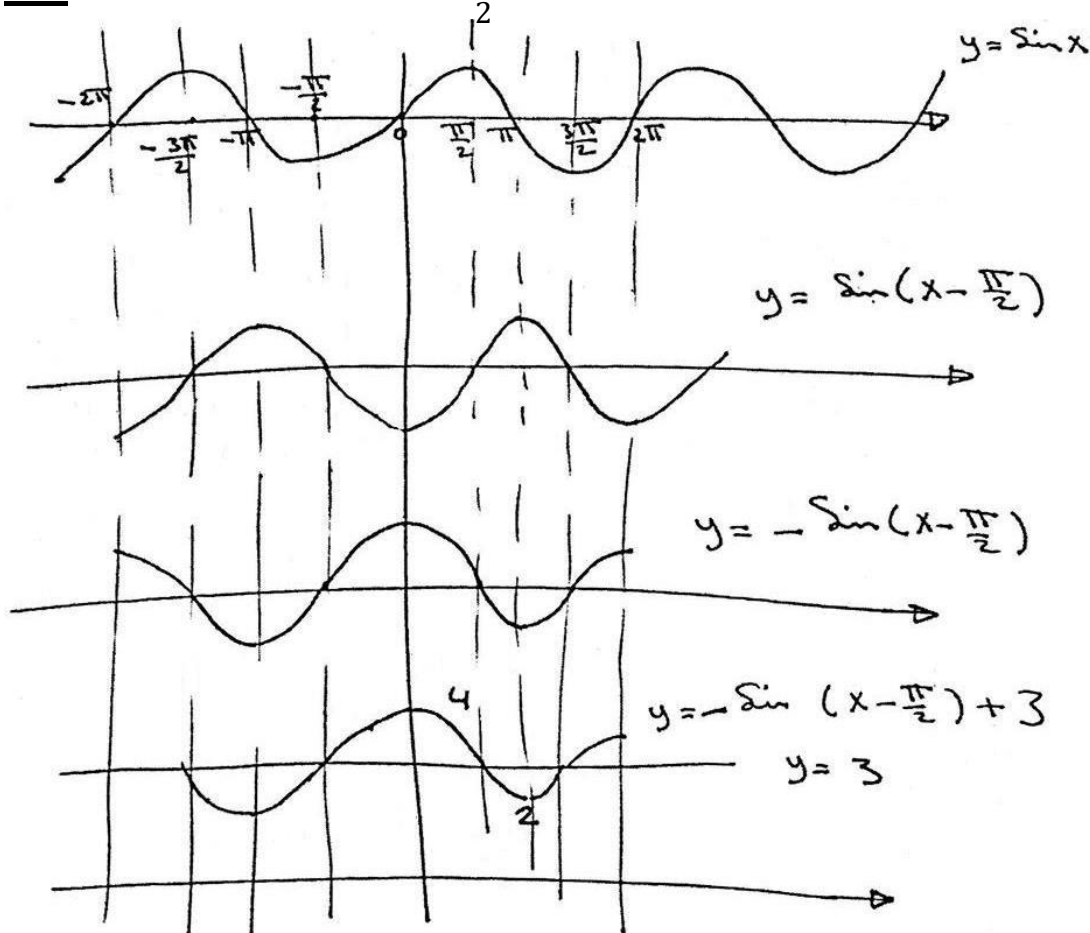


Ex: Graph $y = x^2 + 2x + 3$

$$y = x^2 + 2x + 1 - 1 + 3 \Rightarrow y = (x + 1)^2 + 2$$



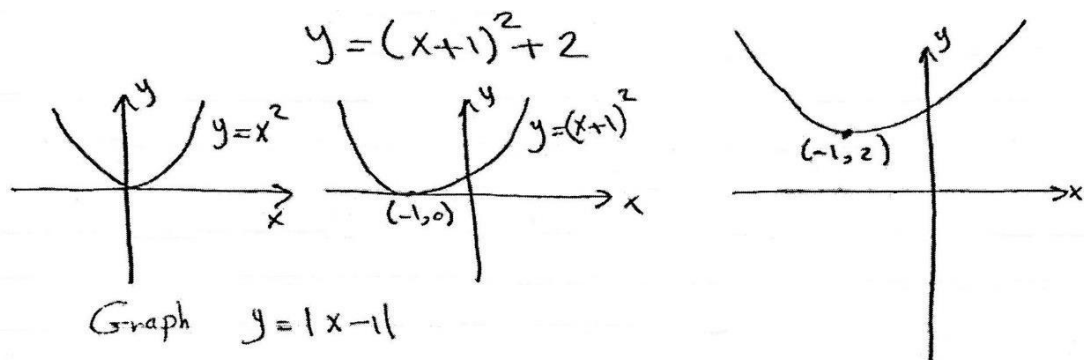
Ex: Graph $y = -\sin(x - \frac{\pi}{2}) + 3$



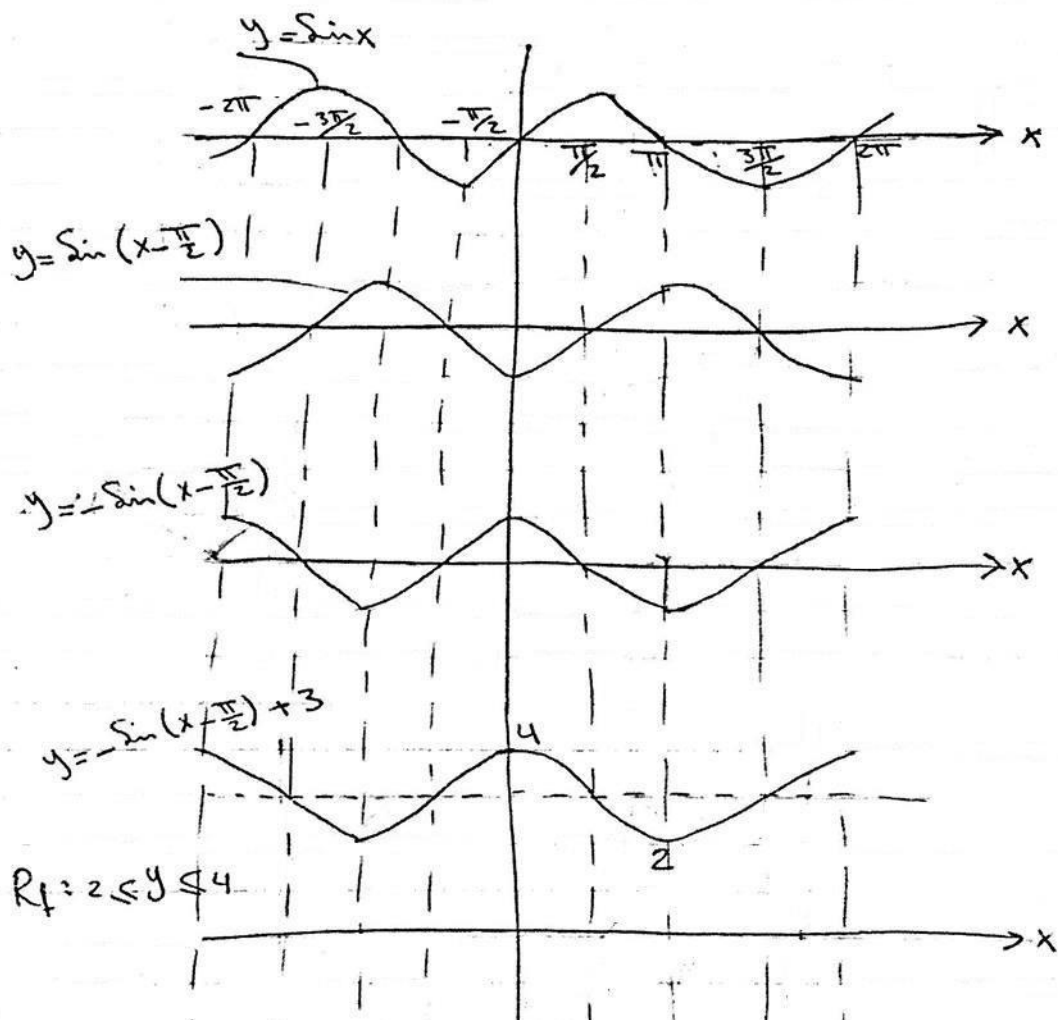
$$R_f: 2 \leq y \leq 4$$

Ex: Graph $y = x^2 + 2x + 3 = x^2 + 2x + 1 - 1 + 3$

$$y = (x + 1)^2 + 2$$



Ex: Graph $y = -\sin\left(x - \frac{\pi}{2}\right) + 3$



Q1: A body moves in a straight line according to the law of motion

$$s = t^3 - 4t^2 - 3t$$

Find it's acceleration at each instant when the velocity is zero.

$$v = \frac{ds}{dt} = 3t^2 - 8t - 3, a = \frac{dv}{dt} = 6t - 8$$

$$0 = 3t^2 - 8t - 3 \Rightarrow 0 = (3t + 1)(t - 3)$$

$$\therefore t = -\frac{1}{3} \text{ or } t = 3$$

$$\therefore a = -10 \text{ or } a = 10$$

Q2: Find the velocity $v = \frac{ds}{dt}$ and acceleration $a = \frac{dv}{dt}$

$$1) s = 2t^3 - 5t^2 + 4t - 3$$

$$2) s = gt^2/2 + v_0 t + s_0, (g, v_0, s_0 \text{ constants})$$

$$3) s = (2t + 3)^2$$

Q3: Find y' and y''

$$1) y = 2x^4 - 4x^2 - 8$$

$$2) 2y = 6x^4 - 18x^2 - 12x$$

$$3) y = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x + 3$$

$$4) y = (3x - 1)(2x + 5)$$

$$5) y = \sin 3x \cdot \tan^2(x + 2)^2 \cdot \sqrt{x - 1}$$

$$6) y = \frac{1}{\sec(x^2y+h)} \text{ where h is constant}$$

(3) Integration

$$1) \int dx = x + c$$

$$2) \int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$$

$$3) \int k \cdot f(x) dx = k \int f(x) dx \text{ where } k \text{ is constant}$$

$$4) \int x^n dx = \frac{x^{n+1}}{n+1} + c \quad n \neq -1$$

$$5) \int \frac{dx}{x} = \ln |x| + c$$

Ex: Evaluate the following integrals

$$1) \int (x^2 + x) dx = \int x^2 dx + \int x dx = \frac{x^3}{3} + \frac{x^2}{2} + c$$

$$2) \int 4x dx = 4 \int x dx = 4 \frac{x^2}{2} + c = 2x^2 + c$$

$$3) \int (x^2 + 1)^5 \cdot 2x dx = \frac{(x^2+1)^6}{6} + c$$

$$\begin{aligned} 4) \int (x^3 + 3x + 5)(x^2 + 1) dx \\ = \int (x^3 + 3x + 5)(x^2 + 1) \frac{3}{3} dx \\ = \frac{1}{3} \frac{(x^3+3x+5)^2}{2} + c \end{aligned}$$

$$\begin{aligned} 5) \int \frac{x^2+x}{7} dx &= \frac{1}{7} \int x^2 dx + \frac{1}{7} \int x dx \\ &= \frac{x^3}{21} + \frac{x^2}{14} + c \end{aligned}$$

$$6) \int \frac{dx}{x^5} = \int x^{-5} dx = \frac{x^{-4}}{-4} + c = -\frac{1}{4x^4} + c$$

3-1 Integration of trigonometric function

Since $\frac{d}{dx}(\sin x) = \cos x$ then $\int \cos x \, dx = \sin x + c$

Similarly for other trigonometric function

$$1) \int \cos u \, du = \sin u + c$$

$$2) \int \sin u \, du = -\cos u + c$$

$$3) \int \sec^2 u \, du = \tan u + c$$

$$4) \int \csc^2 u \, du = -\cot u + c$$

$$5) \int \sec u \tan u \, du = \sec u + c$$

$$6) \int \csc u \cot u \, du = -\csc u + c$$

$$7) \int \tan u \, du = \int \frac{\sin u}{\cos u} dx = -\ln |\cos x| + c$$

$$8) \int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \ln |\sin x| + c$$

$$9) \int \sec x \, dx = \ln |\sec x + \tan x| + c$$

$$10) \int \csc x \, dx = \ln |\csc x - \cot x| + c$$

Ex:

$$1) \int \cos 3x \, dx = \frac{1}{3} \int \cos 3x * 3x dx = \frac{1}{3} \sin 3x + c$$

$$2) \int \sin 7x \, dx = -\frac{1}{7} \cos 7x + c$$

$$3) \int \sec^2(x + 3) dx = \tan(x + 3) + c$$

$$\begin{aligned} 4) \int \cot 2x \cdot \csc 2x \, dx &= -\frac{1}{2} \int -\csc 2x \cot 2x * 2 \, dx \\ &= -\frac{1}{2} \csc 2x + c \end{aligned}$$

$$\begin{aligned} 5) \int x \sin(2x^2) \, dx &= \frac{1}{4} \int \sin(2x^2) * 4x \, dx \\ &= -\frac{1}{4} \cos 2x^2 + c \end{aligned}$$

$$\begin{aligned} 6) \int 2 \sin x \cos x \, dx &= 2 \int (\sin x) \cos x \, dx \\ &= 2 \frac{\sin^2 x}{2} + c = \sin^2 x + c \end{aligned}$$

Definite integrals:-

$$\int_a^b f(x) \, dx$$

a is called the lower bound

b is called the upper bound

properties of definite integrals:-

$$1) \int_a^a f(x) \, dx = 0$$

Ex: $\int_3^3 \frac{3x^3+3x^2-2}{\cos^3 x} \, dx = 0$

$$2) \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$$

Ex: $\int_1^3 2x \, dx = -2 \int_3^1 x \, dx$

$$3) \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$$

Ex: $\int_1^3 2x \, dx = \int_1^2 2x \, dx + \int_2^3 2x \, dx$

$$4) \int_a^b k f(x) dx = k \int_a^b f(x) dx$$

$$5) \int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

6) If (f) is even function then

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

Ex: $\int_{-4}^4 (x^2 - 5) dx = 2 \int_0^4 (x^2 - 5) dx$

7) If (f) is odd function then

$$\int_{-a}^a f(x) dx = 0$$

$$\int_{-4}^4 2x dx = 0$$

Find the curve whose slope at the point (x,y) is $3x^2$ if it passes through point (1,-1)

$$\frac{dy}{dx} = 3x^2 \Rightarrow \int dy = \int 3x^2 dx$$

$$y = x^3 + c$$

$$-1 = 1 + c \Rightarrow c = -2$$

$$y = x^3 - 2$$

Find the curve whose slope at the point (x,y) is $x\sqrt{x^2 - 1}$ and passes through point (2,3)

$$y = \int x\sqrt{x^2 - 1} dx \Rightarrow y = \frac{1}{2} \int (x^2 - 1)^{1/2} * 2x$$

$$y = \frac{1}{3}(x^2 - 1)^{3/2} + c$$

$$3 = \frac{1}{3}(4 - 1)^{3/2} + c \Rightarrow 3 = \frac{1}{3}(3)^{3/2} + c$$

$$c = 1.268$$

Find the velocity of acceleration of a particle whose position function is

$$s(t) = \frac{1}{t^2 + 2t + 1} \quad \text{when } t=2$$

Examples:

$$\begin{aligned} 1) \int_{-1}^1 (2x^2 - x^3) dx &= \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_{-1}^1 \\ &= \left(\frac{2}{3} - \frac{1}{4} \right) - \left(\frac{-2}{3} - \frac{1}{4} \right) = \frac{2}{3} - \frac{1}{4} + \frac{2}{3} + \frac{1}{4} = \frac{4}{3} \end{aligned}$$

$$\begin{aligned} 2) \int_{-3}^{-1} \frac{1}{x^2} - \frac{1}{x^3} dx &= \left[-\frac{1}{x} + \frac{1}{2x^2} \right]_{-3}^{-1} \\ &= \left(1 + \frac{1}{2} \right) - \left(\frac{1}{3} + \frac{1}{18} \right) = \frac{10}{9} \end{aligned}$$

$$3) \int_1^4 \frac{dx}{\sqrt{x}} = \int_1^4 x^{-\frac{1}{2}} dx = 2\sqrt{x} \Big|_1^4 = 2(\sqrt{4} - \sqrt{1}) = 2$$

$$\begin{aligned} 4) \int_0^2 (2-x)^2 dx &= \int_0^2 4 - 4x + x^2 dx \\ &= 4x - 2x^2 + \frac{x^3}{3} \Big|_0^2 = (8 - 8 + \frac{8}{3}) = \frac{8}{3} \end{aligned}$$

Evaluate the following integrals:-

$$\begin{aligned} 1) \int \frac{\cos 2x}{\sin^3 2x} dx &= \int \cos 2x * (\sin 2x)^{-3} dx * \frac{2}{2} \\ &= \frac{1}{2} \frac{(\sin 2x)^{-2}}{-2} + c \end{aligned}$$

$$\begin{aligned}\text{Or } \int \frac{\cos 2x}{\sin^3 2x} dx &= \int \frac{\cos 2x}{\sin 2x} * \frac{1}{\sin^2 2x} dx = \int \cot 2x * \csc^2 2x dx \\ &= -\frac{1}{2} \frac{(\cot 2x)^2}{2} + c\end{aligned}$$

$$\begin{aligned}\text{Or } \int \frac{\cos 2x}{\sin^3 2x} &= \int \cot 2x * \csc 2x * \csc 2x dx * \frac{-2}{-2} \\ &= -\frac{1}{2} \frac{(\csc 2x)^2}{2} + c \quad \text{Or} \quad \text{طريقة الفرضية}\end{aligned}$$

$$\text{Let } y = \sin 2x \Rightarrow dy = \cos 2x * 2 dx$$

$$\frac{dy}{2} = \cos 2x * dx$$

$$\int \frac{\cos 2x}{\sin^3 2x} dx = \int \frac{dy/2}{y^3} = \frac{1}{2} \int \frac{dy}{y^3} = \frac{1}{2} \int y^{-3} dy$$

$$= \frac{1}{2} \frac{y^{-2}}{-2} + c = \frac{1}{2} \frac{(\sin 2x)^{-2}}{-2} + c$$

$$\begin{aligned}2) \int \frac{\cos x dx}{\sqrt{1-\sin x}} &= \int \frac{\cos x}{(1-\sin x)^{1/2}} = \int (1-\sin x)^{-\frac{1}{2}} \cos x dx \\ &= -\frac{(1-\sin x)^{1/2}}{1/2} + c\end{aligned}$$

$$\begin{aligned}3) \int \sqrt{1+\sin x} dx * \frac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}} &= \int \frac{\sqrt{1+\sin^2 x}}{\sqrt{1+\sin x}} dx \\ &= \int (1+\sin x)^{-\frac{1}{2}} * \cos x dx = \frac{(1+\sin x)^{1/2}}{1/2} + c\end{aligned}$$

$$\begin{aligned}4) \int \cos^3 3x dx &= \int \cos^2 3x \cdot \cos 3x dx \\ &= \int (1 - \sin^2 3x) * \cos 3x dx \\ &= \int \cos 3x dx - \int \sin^2 3x \cos 3x dx\end{aligned}$$

$$= \frac{1}{3} \sin 3x - \frac{1}{3} \frac{\sin^3 3x}{3} + c$$

Evaluate the following integration:

$$1) \int \frac{(Z+1) dZ}{\sqrt{Z^2+2Z+2}} = \frac{1}{2} \int 2(Z+1)(Z^2+2Z+2)^{-\frac{1}{2}} dZ$$

$$= \frac{1}{2} * \frac{(Z^2+2Z+2)^{\frac{2}{2}}}{\frac{2}{3}} + c$$

$$2) \int \frac{d\theta}{\sin^2 \frac{\theta}{3}} = \int \csc^2 \frac{\theta}{3} d\theta \qquad \int \csc^2 \theta = -\cot \theta$$

$$= -3 \int -\frac{1}{3} \csc^2 \frac{\theta}{3} d\theta = -3 \cot \frac{\theta}{3} + c$$

$$3) \int \frac{\sqrt{\sin^2 x}}{1+\cos x} dx = \int \frac{\sqrt{\sin^2 x}}{\sqrt{1+\cos x}} dx$$

$$= - \int -\sin x (1+\cos x)^{-\frac{1}{2}} dx$$

$$= - \frac{(1+\cos x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$4) \int \frac{\sqrt{\cot x}}{\sin^2 x} dx = - \int (\cot x)^{\frac{1}{2}} * -\csc^2 x dx$$

$$= - \frac{(\cot x)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$5) \int \sin(\tan \theta) \sec^2 \theta d\theta$$

$$\int \sin x dx = -\cos x + c$$

$$= -\cos(\tan \theta) + c$$

$$\frac{1}{2} \int 2x \sin x^2 dx = \frac{1}{2} * -\cos x^2$$

$$6) \int (\tan \theta + \cot \theta)^2 d\theta$$

$$= \int (\tan^2 \theta + 2 \tan \theta \cot \theta + \cot^2 \theta) d\theta$$

$$= \int \tan^2 \theta d\theta + 2 \int d\theta + \int \cot^2 \theta d\theta$$

$$= \int (\sec^2 \theta - 1) d\theta + 2 \int d\theta + \int (\csc^2 \theta - 1) d\theta$$

$$= \tan \theta - \theta + 2\theta - \cot \theta - \theta + c$$

$$= \tan \theta - \cot \theta + c$$

$$\cos 2x = \cos(x + x) = \cos x \cos x - \sin x \sin x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = 1 - 2\sin^2 x \rightarrow \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin(x + x) = \sin x \cos x + \cos x \sin x$$

$$7) \int \sin 2t (\cos^2 t - \sin^2 t) dt$$

$$= \int \sin 2t \cos 2t dt = \frac{1}{2} \int 2(\sin 2t) \cos 2t dt$$

$$= \frac{1}{2} \frac{(\sin 2t)^2}{2} + c$$

$$8) \int \sin^2 \frac{\theta}{2} d\theta = \int \frac{1 - \cos \theta}{2} d\theta$$

$$= \frac{1}{2} \int (1 - \cos \theta) d\theta = \frac{1}{2} \int d\theta - \frac{1}{2} \int \cos \theta d\theta$$

$$= \frac{1}{2} \theta - \frac{1}{2} \sin \theta + c$$

$$9) \int \frac{\tan \theta}{\cos x \sqrt{1 + \sec x}} dx$$

$$\int \sec x \tan x (1 + \sec x)^{-\frac{1}{2}} dx$$

$$= \frac{(1 + \sec x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$10) \int (\sqrt{x} + \frac{1}{\sqrt{x}}) dx = \int x^{\frac{1}{2}} dx + \int x^{-\frac{1}{2}} dx$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

Q1: Evaluate the following integrals:-

$$1) \int \cos^2 x (2 + \sin x) dx = \int 2 \cos^2 x dx + \int \cos^2 x \sin x dx$$

$$= \int 2 \frac{1 + \cos 2x}{2} dx + \int \cos^2 x \sin x dx$$

$$= x + \frac{1}{2} \sin 2x - \frac{\cos^3 2x}{6} + c$$

$$2) \int \sec^3 2x \tan 2x dx = \frac{1}{2} \int \sec^2 2x * \sec 2x * \tan 2x * 2 dx$$

$$= \frac{1}{2} \frac{\sec^3 2x}{3} + c = \frac{\sec^3 2x}{6} + c$$

$$3) \int \sec^2 \frac{x}{3} (1 + \tan^2 \frac{x}{3}) dx = \int \sec^2 \frac{x}{3} dx + \int \sec^2 \frac{x}{3} \tan^2 \frac{x}{3} dx$$

$$= 3 \tan \frac{x}{3} + 3 \frac{\tan^3 x/3}{3} + c = 3 \tan \frac{x}{3} + \tan^3 \frac{x}{3} + c$$

$$4) \int \frac{dx}{\cos^2 x (2 + 3 \tan^2 \frac{x}{2})^4} = \frac{2}{3} \int (2 + 3 \tan^2 \frac{x}{2})^{-4} * \sec^2 \frac{x}{2} dx * 3 * \frac{1}{2}$$

$$= \frac{2}{3} * \frac{(2+3 \tan \frac{x}{2})^{-3}}{-3} = -\frac{2}{9} (2 + 3 \tan \frac{x}{2})^{-3} + c$$

Q2: Find $\frac{dy}{dx}$

$$1) \sin^2 xy + \tan y = x^2 - y \Rightarrow 2 \sin xy \cdot \cos xy \cdot (xy' + y) + \sec^2 y \cdot y' = 2x - y'$$

$$2x \sin xy \cdot \cos xy \cdot y' + 2y \sin xy \cos xy + \sec^2 y \cdot y' = 2x - y'$$

$$y'(2x \sin xy \cos xy) + \sec^2 y + 1 = 2x - 2y \sin xy \cos xy$$

$$y' = \frac{2x - 2y \sin xy \cos xy}{2x \sin xy \cos xy + \sec^2 y + 1}$$

$$2) \sec^3 y + \cot(xy) = x^3 - y^2 \Rightarrow 3 \sec^2 y \cdot \sec y \tan y \cdot y' - \csc^2(xy)(xy' + y) = 3x^2 - 2yy'$$

$$- \csc^2(xy)(xy' + y) = 3x^2 - 2yy'$$

$$\Rightarrow (3 \sec^2 y \cdot \sec y \cdot \tan y - x \csc^2 xy + 2y)y' = y \csc^2 xy + 3x^2$$

$$y' = \frac{y \csc^2 xy + 3x^2}{3 \sec^3 y \cdot \tan y - x \csc^2 xy + 2y}$$

Q3: Calculate $\lim_{x \rightarrow 2} \frac{(x^2 + x - 6) \tan(x-2)}{(x-2)^2}$

$$= \lim_{x \rightarrow 2} \frac{(x-2)(x+3) \frac{\sin(x-2)}{\cos(x-2)}}{(x-2)^2}$$

$$= \lim_{x \rightarrow 2} \frac{(x+3)}{\cos(x-2)} \boxed{\frac{\sin(x-2)}{x-2}} \Rightarrow 1 \quad \therefore \frac{2+3}{\cos(0)} = \frac{5}{1} = 5$$

MATRICES AND DETERMINANTS

المصفوفات و المحددات

1) MATRICES

ت

المصفوفة (matrix) هي مجموعة من الاعداد الحقيقية أو الاعداد المركبة أو العناصر، مرتبة في صفوف (rows) و اعمدة (columns) و بذلك تشكل حيز متسطيل مثل:

$$\begin{bmatrix} 5 & 7 & 2 \\ 6 & 3 & 8 \end{bmatrix} : \text{This is called a } 2 \times 3 \text{ matrix because it has (2) rows}$$

and (3) columns

❖ مرتبة المصفوفة (order) يشير الى عدد صفوفها و اعمدتها و يرمز له $m \times n$ (حيث ان m هي عدد الصفوف و n عدد الاعمدة).

❖ في المصفوفات يتم حصر الارقام (أو العناصر) بين اقواس مربعة كبيرة []

❖ المصفوفة هي حيز من الارقام التي قد لا توجد علاقة رياضية بينهم، و بذلك فإنه لا يوجد ناتج عددي للمصفوفة (اي ان المصفوفة لا تساوي عدد ما وانما هي فقط ترتيب معين من الارقام).

❖ الترميز المزدوج (Double suffix notation)

لكي يتم تمييز موقع عنصر ما في المصفوفة فإنه يتم استخدام الرمز المزدوج مثل a_{23} (حيث ان a هي العنصر (الرقم). والرقم 2) يشير الى الصف الثاني و الرقم 3) يشير الى العمود الثالث و بذلك فإن موقع هذا العنصر يكون في الصف الثاني، العمود الثالث من المصفوفة.

$$\text{Ex: } \begin{bmatrix} 6 & -5 & 1 & -3 \\ 2 & -4 & 8 & 3 \\ -2 & 9 & 7 & -1 \end{bmatrix} \text{ the order is } (3 \times 4)$$

$$\begin{array}{lll} a_{24} = 3 & a_{33} = 7 & a_{22} = -4 \\ a_{14} = -3 & a_{23} = 8 & a_{11} = 6 \end{array} \quad \text{and so on}$$

❖ يمكن ان نعطي رمز للمصفوفة مثل حرف A أو حرف B أو حرف C الخ لكي يتم تمييز مصفوفة عن الاخرى مثلاً :-

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, [B] = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

∴ [A] is a 2*3 matrix and [B] is a 3*2 matrix

❖ *Equal Matrices:* المصفوفات المتساوية

Two matrices are said to be equal if corresponding elements throughout are equal. Therefore, the two matrices must be of the same order.

❖ *Addition And Subtraction* (جمع و طرح المصفوفات)

لكي يتم جمع أو طرح مصفوفتين يجب ان تكونا اولاً من نفس المرتبة (order). فأذا كانتا كذلك فإنه يتم جمع أو طرح العناصر المتقابلة (المتناظرة في الموقع) في المصفوفتين.

Ex: If $A = \begin{bmatrix} 4 & 2 & 3 \\ 5 & 7 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 8 & 9 \\ 3 & 5 & 4 \end{bmatrix}$

$$\therefore A+B = \begin{bmatrix} 4+1 & 2+8 & 3+9 \\ 5+3 & 7+5 & 6+4 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 12 \\ 8 & 12 & 10 \end{bmatrix}$$

$$\text{And } B - A = \begin{bmatrix} 1-4 & 8-2 & 9-3 \\ 3-5 & 5-7 & 4-6 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 6 \\ -2 & -2 & -2 \end{bmatrix}$$

❖ *Multiplication of Matrices* (ضرب المصفوفات)

أ. ضرب المصفوفة برقم ثابت : يضرب الرقم في جميع عناصر المصفوفة.

Ex: $4 * \begin{bmatrix} 3 & 2 & 5 \\ 6 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 12 & 8 & 20 \\ 24 & 4 & 28 \end{bmatrix}$

ب. ضرب مصفوفة في مصفوفة : يمكن ضرب مصفوفة في أخرى فقط اذا توفر الشرط التالي :-
"عدد اعمدة المصفوفة الاولى = عدد صفوف المصفوفة الثانية". و اذا لم يتوفر هذا الشرط فلا يمكن ضرب المصفوفتين.

Ex: $[A]_{3*2}, [B]_{2*4}$

∴ A * B is possible [since (3*2)*(2*4)] the dimensions of the new matrix will be 3*4

$B * A$ is not possible [since $(2*4)*(3*2)$]

لا يتوفر الشرط لذلك لا يمكن ضرب المصفوفتين بهذا الترتيب.

❖ إذا توفر الشرط السابق يمكن ضرب المصفوفتين، و يتم اجراء الضرب كالتالي:

◀ تضرب عناصر الصف الاول (من المصفوفة الاولى) (في عناصر العمود الاول) من المصفوفة الثانية).

ويتم وضع الناتج في مصفوفة جديدة في الموقع a_{11} (حاصل جمع الضرب).

◀ نضرب عناصر الصف الثاني (من المصفوفة الاولى) (في عناصر العمود الاول) من المصفوفة الثانية).

ويتم وضع الناتج في المصفوفة الجديدة في الموقع a_{21} (حاصل جمع الضرب).

◀ نستمر بنفس الاسلوب الى ان تكتمل جميع عناصر المصفوفة الجديدة.

$$\text{Ex: } A = \begin{bmatrix} 4 & 7 & 6 \\ 2 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

بما ان المصفوفة A هي $2*3$ و المصفوفة B هي $3*1$ ، اذا توفر الشرط

$$\therefore [A]*[B] = (2*3)*(3*1)$$

حيث تساوي عدد الاعمدة في المصفوفة الاولى مع عدد الصفوف في المصفوفة الثانية و بذلك سنتنتج مصفوفة جديدة ذات مرتبة $2*1$.

$$[A*B] = \begin{bmatrix} (4)(8) + (7)(5) + (6)(9) \\ (8)(2) + (3)(5) + (1)(9) \end{bmatrix} = \begin{bmatrix} 121 \\ 40 \end{bmatrix}$$

$$\text{Ex: } [A] = \begin{bmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 4 \end{bmatrix}, [B] = \begin{bmatrix} 8 & 4 & 3 & 1 \\ 2 & 5 & 8 & 6 \end{bmatrix}$$

$$\therefore [A * B]$$

$$\begin{aligned}
 & \begin{bmatrix} (8)(1) + (5)(2) & (4)(1) + (5)(5) & (3)(1) + (5)(8) & (1)(1) + (5)(6) \\ (8)(2) + (7)(2) & (4)(2) + (5)(7) & (3)(2) + (7)(8) & (1)(2) + (7)(6) \\ (8)(3) + (4)(2) & (4)(3) + (5)(4) & (3)(3) + (4)(8) & (1)(3) + (4)(6) \end{bmatrix} \\
 & = \begin{bmatrix} 18 & 29 & 43 & 31 \\ 30 & 43 & 62 & 44 \\ 32 & 32 & 41 & 27 \end{bmatrix}
 \end{aligned}$$

)Transpose Matrix(

❖ المصفوفة المحولة

إذا تم تبديل مواقع الأعمدة في مصفوفة ما بمواقع الصفوف لنفس المصفوف أي أن الصف الأول يصبح العمود الأول و الصف الثاني يصبح العمود الثاني و الصف الثالث يصبح العمود الثالث و هكذا.....فإن المصفوفة الجديدة المتشكلة من هذا

التحويل تدعى (Transpose matrix) و يرمز لها بالرمز A^T (إذا كانت المصفوفة الأولية هي A).

$$\text{Ex: If } A = \begin{bmatrix} 4 & 6 \\ 7 & 9 \\ 2 & 5 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} 4 & 7 & 2 \\ 6 & 9 & 5 \end{bmatrix}$$

$$\text{Ex: (H.W) if } A = \begin{bmatrix} 2 & 7 & 6 \\ 3 & 1 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} \text{ then find: } (A * B)^T$$

$$(\text{ans. } \begin{bmatrix} 35 & 20 \\ 79 & 32 \end{bmatrix})$$

❖ Special matrices:

a) Square matrix: is a matrix of order $m \times m$

المصفوفة المربعة: هي المصفوفة التي يتساوي فيها عدد الأعمدة مع عدد الصفوف.

b) Diagonal matrix: is a square matrix with all elements zero except these on the leading diagonal.

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix} \text{ is a diagonal matrix}$$

c) Unit matrix: is a diagonal matrix in which the elements on the leading diagonal are all unity i.e. :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ is a unit matrix and denoted by } I$$

$$\text{Note : } A.I = I.A = A$$

d) Null matrix: a null matrix is a matrix whose elements are all zero

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2) DETERMINANTS

المحددات

With each square matrix A we associate a number called the determinant of $[A]$ (denoted by $\det. A$ or $|a_{ij}|$). The determinant of matrix is calculated from the entries

$(a_{11}, a_{12}, a_{13}, \dots)$ of A .

❖ من كل مصفوفة مربعة (مثل المصفوفة A) (يمكن ان نستخرج رقم يدعى) محددة المصفوفة (و ذلك بأن نعامل عناصر المصفوفة) a_{11}, a_{12}, a_{13} بطريقة معينة لاستخراج هذا المحدد. وبصورة عامة يمكن ايجاد المحدد بطريقة Minors and cofactors.

❖ Minors and cofactors:

a) Minors: the minor of the element a_{ij} [where (i) is the number of row, and (j) is the number of column] is a matrix $[A]$ in the determinant of the matrix which remains after the row and the column containing the element a_{ij} are deleted.

❖ لكل عنصر من عناصر مصفوفة ما (مثل المصفوفة A) (هناك محدد تدعى) minor (لعنصر المعين. هذا ال minor) يمكن ايجاده من خلال حذف الصف و العمود الذين يحتويان العنصر المعين. المحدد المتبقية بعد الحذف تدعى minor (العنصر).

$$\text{Ex: If } [A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then } \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

ملاحظة: يتم تمييز المصفوفة عن المحددة بالاقواس التي تحيط بالعناصر. فالمصفوفة اقواسها مربعة

كبيرة [] ، و المحددة اقواسها خطوط مستقيمة | | .

$$\text{The minor of } a_{11} \text{ is : } \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \quad \text{The minor of } a_{22} \text{ is : } \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\text{The minor of } a_{12} \text{ is : } \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \quad \text{The minor of } a_{23} \text{ is : } \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

and so on.

b) Cofactors: the cofactor of a_{ij} , is the determinant (A_{ij}) that is $(-1)^{i+j}$ times the minor of a_{ij} :-

$$A_{ij} = (-1)^{i+j} * \text{minor of } a_{ij}$$

❖ اعطاء اشارة (سالبة او موجبة) الى (minor) عنصر معين يؤدي الى حصولنا على (cofactor) ذلك العنصر (يرمز له a_{ij}). الاشارة تعتمد على موقع العنصر في المصفوفة و تتبع القاعدة اعلاه أو الجدول ادناه:

$$\begin{array}{ccccc} + & - & + & \cdot & \cdot \\ - & + & - & \cdot & \cdot \\ | + & - & + & \cdot & \cdot | \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Ex: if $[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

➤ The minor of $a_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

The cofactor of a_{31} , $A_{31} = (-1)^{3+1} * \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

➤ The minor of $a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

The cofactor of a_{23} , $A_{23} = (-1)^{2+3} * \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \dots\dots\dots$ And so on.

❖ We can find the value of the determinant by summing the (cofactors*entry) of the entries of any one row or any one column, thus:-

يمكن إيجاد قيمة المحدد لأي مصفوفة مربعة باستخدام أي صف أو أي عمود (ويفضل استخدام الصف أو العمود الذي يحتوي أكبر قدر ممكن من الأصفار لتسهيل العمليات الحسابية)

Ex: $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ ❖ نأخذ الصف الأول مثلاً:

$$\begin{aligned} &= a_{11} \cdot A_{11} + a_{12} \cdot A_{12} + a_{13} \cdot A_{13} \\ &= a_{11} \cdot (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} \cdot (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ &\quad + a_{13} \cdot (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

❖ و إذا تم اختيار العمود الثالث مثلاً:-

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{13} \cdot (-1)^{1+3} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} + a_{23} \cdot (-1)^{2+3} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$+ a_{33} \cdot (-1)^{3+3} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

و هكذا الحال اذا تم اختيار اي عمود أو أي صف.

❖ EXPANDING AND EVALUATING A DETERMINANT:

- For a matrix with only one element, $\det [a] = a$
- For a matrix with 2×2 , $\det \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$
- For a matrix with 3×3

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (a_{11} \cdot a_{22} \cdot a_{33}) + (a_{12} \cdot a_{23} \cdot a_{31}) + (a_{13} \cdot a_{21} \cdot a_{32}) - (a_{31} \cdot a_{22} \cdot a_{13})$$

$$- (a_{32} \cdot a_{23} \cdot a_{11}) - (a_{33} \cdot a_{21} \cdot a_{12})$$

ملاحظة مهمة : هذه الطريقة في فتح المحدد خاصة بمحددة 3×3 و لا يجوز استخدامها مطلقا اذا كانت

المحددة اكبر من 3×3 .

❖ For a matrix more than 3×3 we use the definition of (minor) and (cofactors).

اذا كانت المصفوفة المربعة اكبر من 3×3 فيجب ان نستخدم تعاريف (minor و cofactor) لكي يتم

ايجاد المحددة، لان استخدام (cofactor) يقلل درجة (مرتبة) المحددة.

Ex: Evaluate $A = \begin{vmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$ حل السؤال سيتم باكثر من طريقة

أ) (الاسلوب الاول : باستخدام (minor و cofactor) واختيار الصف الاول:

$$\det A = a_{11} \cdot A_{11} + a_{12} \cdot A_{12} + a_{13} \cdot A_{13}$$

$$= 2 * (-1)^2 \begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} + (1) * (-1)^3 \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} + (3)(-1)^4 \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= 2(-1 + 6) - 1 * (3 + 4) + 3(9 + 2) = 36$$

ب) (الاسلوب الثاني : باستخدام minor) و (cofactor) واختيار العمود الثالث:

$$\det A = a_{13} \cdot A_{13} + a_{23} \cdot A_{23} + a_{33} \cdot A_{33}$$

$$= 3 (-1)^4 \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} + (-2)(-1)^5 \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + (1)(-1)^6 \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = 36$$

ت) (الاسلوب الثالث : باستخدام الطريقة الخاصة للمحدد 3 * 3)

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} =$$

$$= (2)(-1)(1) + (1)(-2)(2) + (3)(3)(3) - (2)(-1)(3) - (3)(-2)(2) - (1)(3)(1)$$

$$= -2 - 4 + 27 + 6 + 12 - 3 = 36$$

Ex: Find the value of (x) if : $\begin{vmatrix} 1 & 1 & -1 \\ x & 1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = 0$

❖ Expanding first row:

$$1. (-1)^2 \cdot \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + (1)(-1)^3 \cdot \begin{vmatrix} x & 2 \\ 1 & -1 \end{vmatrix} + (-1)(-1)^4 \cdot \begin{vmatrix} x & 1 \\ 1 & 2 \end{vmatrix} = 0$$

$$= (-1 - 4) - (-x - 2) - (2x - 1) = 0$$

$$-5 + x + 2 - 2x + 1 = 0 \rightarrow x = -2$$

Or :

$$\begin{vmatrix} 1 & 1 & -1 & 1 & 1 \\ x & 1 & 2 & x & 1 \\ 1 & 2 & -1 & 1 & 2 \end{vmatrix} = 0$$

$$= (1*1*-1 + 1*2*1 + -1*x*2) - (-1*1*1 + 1*2*2 + 1*x*-1) = 0$$

$$-1 + 2 - 2x + 1 - 4 + x = 0 \rightarrow x = -2$$

❖ **CRAMER'S RULE**

Is a rule for solving a system of linear equations, like three equations with three unknowns

If $a_{11} \cdot x + a_{12} \cdot y + a_{13} \cdot Z = b_1 \dots\dots\dots (1)$

$$a_{21} \cdot x + a_{22} \cdot y + a_{23} \cdot Z = b_2 \dots\dots\dots (2)$$

$$a_{31} \cdot x + a_{32} \cdot y + a_{33} \cdot Z = b_3 \dots\dots\dots (3)$$

Then cramer's rule gives:-

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad Z = \frac{D_z}{D}$$

Where ; $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, $D_x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$

$$D_y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad D_z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Ex: Find the values of x , y and Z by cramer's rule for:

$$y - Z = 3$$

$$2x - 2Z = 2$$

$$2y + Z = 3$$

Sol: Re- arranged the equation as

$$0 \cdot x + 1 \cdot y - 1 \cdot Z = 3$$

$$2 \cdot x + 0 \cdot y - 2 \cdot Z = 2$$

$$0 \cdot x + 2 \cdot y + 1 \cdot Z = 3$$

$$\therefore D = \begin{vmatrix} 0 & 1 & -1 \\ 2 & 0 & -2 \\ 0 & 2 & 1 \end{vmatrix} = -6, \quad D_x = \begin{vmatrix} 3 & 1 & -1 \\ 2 & 0 & -2 \\ 3 & 2 & 1 \end{vmatrix} = \text{zero}$$

$$D_y = \begin{vmatrix} 0 & 3 & -1 \\ 2 & 2 & -2 \\ 0 & 3 & 1 \end{vmatrix} = -12, \quad D_z = \begin{vmatrix} 0 & 1 & 3 \\ 2 & 0 & 2 \\ 0 & 2 & 3 \end{vmatrix} = 6$$

$$\therefore x = \frac{D_x}{D} = \frac{0}{-6} = 0, \quad y = \frac{D_y}{D} = \frac{-12}{-6} = 2, \quad z = \frac{D_z}{D} = \frac{6}{-6} = -1$$

EX: solve the following system of equation by cramer's rule;

$$(t-1).x + t.y = 1$$

$$t.x + (t-1).y = -1$$

$$D = \begin{vmatrix} t-1 & t \\ t & t-1 \end{vmatrix} = (t-1)^2 - t^2 = t^2 - 2t + 1 - t^2 = -2t + 1$$

$$D_x = \begin{vmatrix} 1 & t \\ -1 & t-1 \end{vmatrix} = t-1 + t = 2t-1 \Rightarrow x = \frac{D_x}{D} = \frac{2t-1}{-2t+1} = -1$$

$$D_y = \begin{vmatrix} t-1 & 1 \\ t & -1 \end{vmatrix} = -t+1-t = -2t+1 \Rightarrow y = \frac{D_y}{D} \Rightarrow$$

$$y = \frac{-2t+1}{-2t+1} = 1$$

(H.W) Find the values of (x) if: $\begin{vmatrix} x & 2 & 3 \\ 2 & x+3 & 6 \\ 3 & 4 & x+6 \end{vmatrix} = 0$

Transcendental functions

1) Logarithmic functions

a) Natural logarithmic functions:-

$$\log_e m = x \text{ then } m = e^x$$

where e is a real number ($e = 2.718$)

$$\log_e m = x \text{ shall be written as } \ln m = x$$

b) Common logarithmic functions:-

$$\log_{10} m = x \text{ then } m = 10^x$$

$$\log_{10} m = x \text{ shall always written as } \log m = x$$

$$\therefore \log_{10} m = \log m = x$$

c) General logarithmic functions:-

$$\log_a m = x \text{ then } m = a^x$$

Properties of logarithmic functions:-

$$1. \log(x.y) = \log x + \log y$$

$$2. \ln(x.y) = \ln x + \ln y$$

$$3. \log \frac{x}{y} = \log x - \log y$$

$$4. \log \left(\frac{1}{x} \right) = -\log x = \log x^{-1}$$

$$5. \log x^b = b \log x \quad \left\{ \begin{array}{l} \log 3^2 = 2 \log 3 \\ \log \sin^2 x = 2 \log \sin x \\ \log \sin x^2 \neq 2 \log \sin x \end{array} \right.$$

$$6. \log_a x = \frac{\ln x}{\ln a} \text{ where } a \neq 1$$

$$7. \log_{10} x = \frac{\ln x}{\ln 10}$$

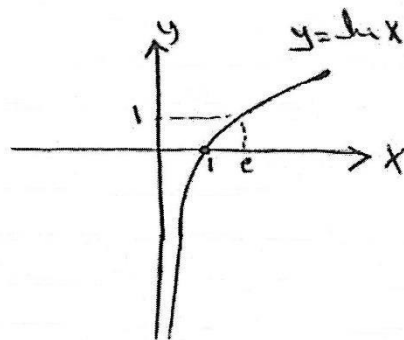
$$8. \log 1 = 0$$

$$9. \log_a a = 1$$

Graph of logarithmic function

ln

- إذا كان الـ مقدار سالب فالنتيجة غير معرفة 1)
 إذا كان الـ x أقل من 1 فالناتج يكون سالب 2) أكبر من
 وإذا كان الـ x 1 فالناتج يكون موجب



Differentiations of logarithmic functions

$$\text{If } y = \log_a u(x) \text{ then } \frac{dy}{dx} = \frac{1}{\ln a} * \frac{1}{u(x)} * \frac{d(u(x))}{dx}$$

Integration of logarithmic functions:-

$$\int \frac{u'(x)}{u(x)} = \ln|u(x)| + c$$

$$\int \frac{y'}{y} = \ln|y| + c$$

$$\int \frac{dx}{x} = \ln|x| + c$$

$$\int \frac{x dx}{x^2 + 1} = \frac{1}{2} \int \frac{2x dx}{x^2 + 1} = \frac{1}{2} \ln|x^2 + 1| + c$$

$$\int \frac{dx}{x \ln x} = \int \frac{\frac{1}{x} dx}{\ln x} = \ln|\ln x| + c$$

Find $\frac{dy}{dx}$:-

$$1) y = \ln x^3 \rightarrow y' = \frac{1}{x^3} * 3x^2 = \frac{3}{x}$$

$$\text{Or } y = 3 \ln x \Rightarrow y' = 3 * \frac{1}{x}$$

$$2) y = \log_5(x^3 - x) \Rightarrow y' = \frac{1}{\ln 5} * \frac{1}{x^3 - x} * (3x^2 - 1)$$

$$3) y = \ln(\ln \sin x) \Rightarrow y' = \frac{1}{\ln(\sin x)} * \frac{1}{\sin x} * \cos x$$

$$4) y = x^{\cos x} \quad \ln \text{ الطرفين } \quad \text{(عندما الاس متغير نأخذ)}$$

$$\ln y = \ln x^{\cos x} \Rightarrow \ln y = \cos x \cdot \ln x$$

$$\frac{1}{y} * y' = \cos x * \frac{1}{x} + \ln x * -\sin x$$

$$y' = x^{\cos x} \left[\frac{\cos x}{x} - \sin x \ln x \right]$$

$$5) y = \sqrt[3]{\frac{(x+1)^2(x-3)^5}{(x^3+1)\cos x}} \Rightarrow y = \left[\frac{(x+1)^2(x-3)^5}{(x^3+1)\cos x} \right]^{\frac{1}{3}}$$

$$\ln y = \ln \left[\frac{(x+1)^2(x-3)^5}{(x^3+1)\cos x} \right]^{\frac{1}{3}}$$

$$\ln y = \frac{1}{3} [\ln(x+1)^2(x-3)^5 - \ln(x^3+1)\cos x]$$

$$\ln y = \frac{1}{3} \ln(x+1)^2 + \frac{1}{3} \ln(x-3)^5 - \frac{1}{3} \ln(x^3+1) - \frac{1}{3} \ln \cos x$$

$$\ln y = \frac{2}{3} \ln(x+1) + \frac{5}{3} \ln(x-3) - \frac{1}{3} \ln(x^3+1) - \frac{1}{3} \ln \cos x$$

$$\frac{y'}{y} = \frac{2}{3} \frac{1}{x+1} + \frac{5}{3} \frac{1}{x-3} - \frac{1}{3} \frac{3x^2}{x^3+1} - \frac{1}{3} \frac{1}{\cos x} * -\sin x$$

$$y' = \dots\dots\dots$$

$$6) y = \sqrt[x]{\cos x} \Rightarrow y = (\cos x)^{\frac{1}{x}} \Rightarrow \ln y = \ln(\cos x)^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x} \ln \cos x$$

$$\frac{y'}{y} = \left[\frac{1}{x} * \frac{1}{\cos x} * -\sin x + \ln \cos x * -\frac{1}{x^2} \right]$$

$$y' = x\sqrt{\cos x} \left[\frac{1}{x} * \frac{1}{\cos x} * -\sin x + \ln \cos x * -\frac{1}{x^2} \right]$$

Evaluate the following integrals:

$$1) \int \frac{\sin x \, dx}{3 \cos x + 5} = -\frac{1}{3} \int \frac{-3 \sin x \, dx}{3 \cos x + 5} = -\frac{1}{3} \ln|3 \cos x + 5| + c$$

$$2) \int \frac{\cos x \, dx}{\sqrt{\sin x}} = \int (\sin x)^{-\frac{1}{2}} \cos x \, dx = \frac{(\sin x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$3) \int \frac{\cot x \, dx}{\log_5 \sin x} = \int \frac{\cot x \, dx}{\frac{\ln \sin x}{\ln 5}} = \ln 5 \int \frac{\cot x \, dx}{\ln \sin x} = \ln 5 \ln|\ln \sin x| + c$$

$$4) \int \frac{2x-5}{x} dx = \int \frac{2x}{x} dx - \int \frac{5}{x} dx = 2x - 5 \ln|x| + c$$

$$5) \int \frac{dx}{x(\ln x)^3} = \int (\ln x)^{-3} \cdot \frac{1}{x} dx = \frac{(\ln x)^{-2}}{-2} + c$$

$$6) \int \frac{dx}{\sqrt{x}(1+\sqrt{x})} = 2 \int \frac{x^{-\frac{1}{2}}}{1+\sqrt{x}} = 2 \ln|1 + \sqrt{x}| + c$$

$$7) \int \frac{dx}{\tan x (\ln \tan x) \cos^2 x} = \int \frac{\frac{1}{\tan x} \cdot \sec^2 x \, dx}{\ln(\tan x)} = \ln|\ln \tan x| + c$$

Ex: Find the values of (x) if

$$2 \ln \sqrt{x-1} + \ln(x+3) = \ln 5$$

$$\textbf{Solve: } \ln(\sqrt{x-1})^2 + \ln(x+3) = \ln 5$$

$$\ln(x-1)(x+3) = \ln 5$$

$$(x-1)(x+3) = 5$$

$$x^2 + 2x - 3 - 5 = 0 \Rightarrow x^2 + 2x - 8 = 0$$

$$(x-2)(x+4) = 0 \text{ either } x = 2 \text{ or } x = -4$$

Exponential function:

A) $y = e^x$ is called the exponential function, where (e) is the exponential number ($e = 2.718$)

Notice that if $y = e^x$ then $\ln y = \ln e^x = x \ln e = x$

$$\ln e^x = e^{\ln x} = x$$

Properties of (e^x)

$$1) e^{x_1+x_2} = e^{x_1} \cdot e^{x_2}$$

$$2) e^0 = 1$$

$$3) e^{-x} = \frac{1}{e^x}$$

$$4) \ln e^x = x$$

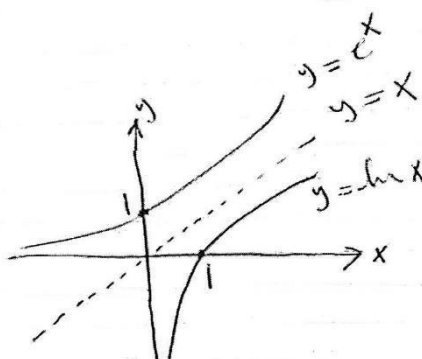
$$5) e^{\ln x} = x$$

Graph of $y = e^x$

Diff of exp. Function

$$\text{If } y = e^{f(x)} \text{ then } \frac{dy}{dx} = e^{f(x)} \cdot f'(x)$$

المشتقة = الاساس $\ln$ * الدالة الاسية * المشتقة للاس



Integration of Exp. function

Since $d(e^x) = e^x \cdot dx$

$$\therefore \int e^x dx = e^x + c$$

Or in general

$$\int e^u \left(\frac{du}{dx} \right) dx = e^u + c$$

إذا توفرت المشتقة للاس في التكامل فالناتج هو الدالة الاصلية

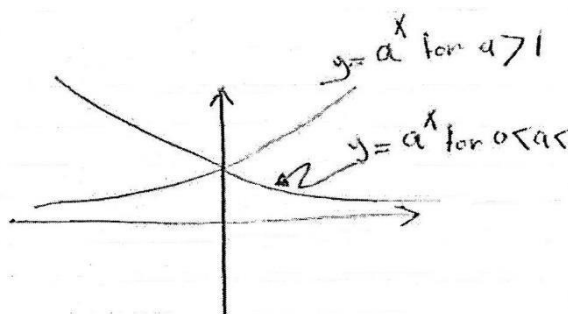
Ex: $\int e^{\sin x} \cdot \cos x \, dx = e^{\sin x} + c$

B) $y = a^x$ is called the exponential function to base (a) where $a > 0, a \neq 1$

Notice that if $y = a^x$ then $y = e^{x \ln a}$

Graph of $y = a^x$

Differentiation of $y = a^x$



If $y = a^x$ where $a > 0$ $a \neq 1$

Then $\frac{dy}{dx}(a^{f(x)}) = \ln a * a^{f(x)} * f'(x)$

Integration of $y = a^x$

$$\int a^{f(x)} \cdot f'(x) = \frac{1}{\ln a} * a^{f(x)} + c$$

Ex: Simplify the following:-

1) $e^{\ln(x^2-1)} = x^2 - 1 = (x - 1)(x + 1)$

2) $e^{\ln 3 + 2 \ln 4} = e^{\ln 3 + \ln 4^2} = e^{\ln 3} * e^{\ln 4^2} = 3 * 4^2 = 48$

3) $\ln \frac{e^{3x}}{5} = \ln e^{3x} - \ln 5 = 3x \ln e - \ln 5 = 3x - \ln 5$

4) $\ln(x^2 * e^{-2x}) = \ln x^2 + \ln e^{-2x} = \ln x^2 - 2x = 2 \ln x - 2x$

5) $e^{2x - \ln x} = \frac{e^{2x}}{e^{\ln x}} = \frac{e^{2x}}{x}$

Ex: Solve the following (find x):

1) $3^x = 2^{x+1} \Rightarrow \ln 3^x = \ln 2^{x+1} \Rightarrow x \ln 3 = (x + 1) \ln 2$

$$x \ln 3 = x \ln 2 + \ln 2$$

$$x \ln 3 - x \ln 2 = \ln 2 \Rightarrow x(\ln 3 - \ln 2) = \ln 2$$

$$x \ln \frac{3}{2} = \ln 2 \Rightarrow x = \frac{\ln 2}{\ln \frac{3}{2}}$$

2) $x^x = 2^x$ for $x > 0$

$$x \ln x = x \ln 2 \Rightarrow x(\ln x - \ln 2) = 0$$

$$\therefore \ln x - \ln 2 = 0 \Rightarrow \ln x = \ln 2 \Rightarrow x = 2$$

3) $4^{-x} = 3^{x+2} \Rightarrow \ln 4^{-x} = \ln 3^{x+2} \Rightarrow -x \ln 4 = (x + 2) \ln 3$

$$-x \ln 4 = x \ln 3 + 2 \ln 3$$

$$-x(\ln 4 + \ln 3) = 2 \ln 3 \Rightarrow x = -\frac{\ln 9}{\ln 12}$$

$$4) \ln(x + \sqrt{x^2 - 1}) = 1$$

$$x + \sqrt{x^2 - 1} = e^1$$

نأخذ e للطرفين

$$\sqrt{x^2 - 1} = e - x$$

نربع الطرفين

$$x^2 - 1 = e^2 - 2ex + x^2$$

$$2ex = e^2 + 1 \Rightarrow x = \frac{e^2 + 1}{2e}$$

$$5) 3^x + 3^{-x} = 4$$

نضرب طرفي المعادلة 3^x

$$3^{2x} + 1 = 4 * 3^x$$

$$3^{2x} - 4 * 3^x + 1 = 0$$

$$3^x = \frac{4 \pm \sqrt{16 - 4}}{2} \Rightarrow 3^x = 2 \pm \sqrt{3}$$

$$x \ln 3 = \ln(2 \pm \sqrt{3})$$

$$x = \frac{\ln(2 \pm \sqrt{3})}{\ln 3}$$

$$6) \log_5(x + 2) + \log_5(x - 2) = 1$$

$$\log_5(x + 2)(x - 2) = 1$$

$$5^1 = (x + 2)(x - 2)$$

$$x^2 - 9 = 0 \Rightarrow x = \pm 3$$

Find $\frac{dy}{dx}$

$$1) y = e^{\sin x} \Rightarrow y' = e^{\sin x} \cdot \cos x$$

$$2) y = 2^{\csc x} \Rightarrow y' = 2^{\csc x} * \ln 2 * -\csc x \cot x$$

$$3) \tan y = e^x + \ln x$$

$$\sec^2 y * y' = e^x + \frac{1}{x} \Rightarrow y' = \dots\dots\dots$$

$$4) y = 2^x \cdot 3^x \Rightarrow \ln y = \ln(2^x \cdot 3^x)$$

$$\ln y = \ln 2^x + \ln 3^x$$

$$\ln y = x \ln 2 + x \ln 3 = x(\ln 2 + \ln 3) = x \ln 6$$

$$\frac{y'}{y} = \ln 6 \Rightarrow y' = 2^x \cdot 3^x \cdot \ln 6$$

$$5) 3^{1+e^y} = 2^{3+4^x}$$

$$3^{1+e^y} * \ln 3 * e^y * \frac{dy}{dx} = 2^{3+4^x} * \ln 2 * 4^x * \ln 4$$

$$6) y = e^{\sin^3 e^{x^2}} \Rightarrow y' = e^{\sin^3 e^{x^2}} * 3 \sin^2 e^{x^2} * \cos e^{x^2} * e^{x^2} * 2x$$

$$7) \ln yx = x^{y^2 \sin x} \Rightarrow \ln(\ln yx) = y^2 \sin x \ln x$$

$$\frac{1}{\ln yx} * \frac{1}{yx} * (x \cdot y' + y) = y^2 \sin x \cdot \frac{1}{x} + \ln x (2yy' \sin x + y^2 \cos x)$$

$$8) y = x^{\sin x} + x^2 \quad \text{لا يمكن اخذ } \ln \text{ الطرفين مباشرة وذلك لكون الطرف الايمن ليس دالة منفصلة}$$

$$y = e^{\ln x^{\sin x}} + x^2$$

$$y = e^{\sin x \ln x} + x^2$$

$$y' = e^{\sin x \ln x} \left(\sin x * \frac{1}{x} + \cos x \cdot \ln x \right) + 2x$$

$$\therefore y' = x^{\sin x} \left(\sin x * \frac{1}{x} + \cos x \cdot \ln x \right) + 2x$$

$$9) y = (x^2 + 1)^{x^3} + \sin(x^x) + x^4$$

$$y = e^{\ln(x^2+1)^{x^3}} + \sin(e^{\ln x^x}) + x^4$$

$$= e^{x^3 \ln(x^2+1)} + \sin(e^{x \ln x}) + x^4$$

$$y' = e^{x^3 \ln(x^2+1)} \left[x^3 * \frac{2x}{x^2 + 1} + \ln(x^2 + 1) * 3x^2 \right] \\ + \cos(e^{x \ln x}) \left[e^{x \ln x} \left(x * \frac{1}{x} + \ln x \right) \right] + 4x^3$$

$$10) y = \tan x^x \Rightarrow y = \tan e^{\ln x^x} = \tan e^{x \ln x}$$

$$y' = \sec^2(e^{x \ln x}) * e^{x \ln x} * \left(x * \frac{1}{x} + \ln x \right)$$

Ex: Evaluate the following integrals:-

$$1) \int (e^x + 2) dx = \int e^x dx + 2 \int dx = e^x + 2x + c$$

$$2) \int \frac{e^{2x} dx}{3e^{2x}+1} = \frac{1}{6} \int \frac{6e^{2x} dx}{3e^{2x}+1} = \frac{1}{6} \ln |3e^{2x} + 1| + c$$

$$3) \int e^{\ln \sqrt{x+1}} dx = \int \sqrt{x+1} dx = \frac{(x+1)^{3/2}}{3/2} + c$$

$$4) \int 3^{\tan 7x} \cdot \sec^2 7x dx = \frac{1}{7 \ln 3} 3^{\tan 7x} + c$$

$$5) \int 2^x \cdot \cos 2^x dx = \frac{1}{\ln 2} \int 2^x * \ln 2 * \cos 2^x dx = \frac{1}{\ln 2} * \sin 2^x + c$$

$$6) \int x^2 \cot(2 + x^3) dx = \int x^2 \frac{\cos(2+x^3)}{\sin(2+x^3)} dx = \frac{1}{3} \ln |\sin(2+x^3)| + c$$

$$7) \int e^{2x} \sec^2(e^{2x} + 1) dx = \frac{1}{2} \tan(e^{2x} + 1) + c$$

$$8) \int \frac{\ln x}{x} dx = \frac{(\ln|x|)^2}{2} + c$$

$$9) \int \frac{\ln(\sin x)}{\tan x} dx = \int (\ln \sin x) \frac{\cos x}{\sin x} dx = \frac{1}{2} (\ln(\sin x))^2 + c$$

$$10) \quad \int \frac{(x-1) dx}{3x^2-6x+5} = \frac{1}{6} \ln |3x^2 - 6x + 5| + c$$

$$11) \quad \int \frac{\sec^3 x + e^{\sin x}}{\sec x} dx = \int \frac{\sec^3 x}{\sec x} dx + \int \frac{e^{\sin x}}{\sec x} dx$$

$$= \int \sec^2 x dx + \int e^{\sin x} * \cos x dx = \tan x + e^{\sin x} + c$$

$$12) \quad \int_1^{e^2} \frac{2 \ln x}{x} dx = \frac{2(\ln x)^2}{2} \Big|_1^{e^2} = (\ln e^2)^2 - (\ln 1)^2 = (2 \ln e)^2 = 4 - 0 = 4$$

$$13) \quad \int e^{\sin^2 x} \cdot \sin 2x dx = \int e^{\sin^2 x} * 2 \sin x \cos x dx = e^{\sin^2 x} + c$$

$$14) \quad \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2e^{\sqrt{x}} + c$$

6 Inverse trigonometric functions

The inverse trigonometric functions are defined as follows

$$y = \sin^{-1} x \Rightarrow x = \sin y$$

$$y = \cos^{-1} x \Rightarrow x = \cos y$$

$$y = \tan^{-1} x \Rightarrow x = \tan y$$

$$y = \sec^{-1} x \Rightarrow x = \sec y$$

$$y = \cot^{-1} x \Rightarrow x = \cot y$$

$$y = \csc^{-1} x \Rightarrow x = \csc y$$

Ex: If $\sin^{-1} \frac{1}{2} = x \therefore \sin x = \frac{1}{2}$ but $\sin \frac{\pi}{6} = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$

Inverse trigonometric identities:-

$$1) \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

$$2) \sin^{-1}(\sin \theta) = \theta$$

$$3) \sin(\sin^{-1}\theta) = \theta$$

$$4) \cos^{-1}(-x) = \pi - \cos^{-1}(x)$$

$$5) \sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$$

$$6) \csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$$

$$7) \sec^{-1}(-x) = \pi - \sec^{-1}(x)$$

Derivative inverse trigonometric function:-

$$1) d(\sin^{-1}u) = \frac{du}{\sqrt{1-u^2}}$$

$$2) d(\cos^{-1}u) = -\frac{du}{\sqrt{1-u^2}}$$

$$3) d(\tan^{-1}u) = \frac{du}{1+u^2}$$

$$4) d(\cot^{-1}u) = -\frac{du}{1+u^2}$$

$$5) d(\sec^{-1}u) = \frac{du}{|u|\sqrt{u^2-1}}$$

$$6) d(\csc^{-1}u) = -\frac{du}{|u|\sqrt{u^2-1}}$$

Integration leading to inverse trigonometric functions

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c$$

$$\int \frac{dx}{1+x^2} = \tan^{-1}x + c$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + c$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

Examples

A) Show that

1) $\sin^{-1}(x)$ is an odd function

$$\text{Let } y = \sin^{-1}(-x) \Rightarrow -x = \sin y$$

$$-y = \sin^{-1}x \Rightarrow y = -\sin^{-1}x$$

$$\therefore \sin^{-1}(-x) = -\sin^{-1}(x) \text{ odd function}$$

B) Evaluate $\cos(\sin^{-1}0.8)$

$$\cos A = \sqrt{1 - \sin^2 A}$$

$$\begin{aligned} \cos(\sin^{-1}0.8) &= \sqrt{1 - \sin^2(\sin^{-1}0.8)} \\ &= \sqrt{1 - \sin(\sin^{-1}0.8) \cdot \sin(\sin^{-1}0.8)} \\ &= \sqrt{1 - (0.8)(0.8)} = \sqrt{1 - 0.64} = 0.6 \end{aligned}$$

C) Find $\frac{dy}{dx}$

$$1) y = \sin^{-1}(x^2) \Rightarrow y' = \frac{2x}{\sqrt{1-x^4}}$$

$$2) y = \cot^{-1}(x \sin x) \Rightarrow y' = \frac{-(x \cos x + \sin x)}{1 + (x \sin x)^2}$$

$$3) \sec^{-1}(x\sqrt{y}) = \sin^{-1} \left(\frac{3}{\ln x} \right)$$

$$\frac{x \cdot \frac{y'}{2\sqrt{y}} + \sqrt{y}}{|x\sqrt{y}|\sqrt{x^2y-1}} = \frac{-3 * \frac{1}{(\ln x)^2}}{\sqrt{1 - \left(\frac{3}{\ln x}\right)^2}}$$

$$4) \sin^{-1}(x^y) + (\cos^{-1}y)^{\sin^{-1}x} = x^2$$

$$\sin^{-1}(e^{\ln x^y}) + e^{\ln(\cos^{-1}y)^{\sin^{-1}x}} = x^2$$

$$\sin^{-1}(e^{y \ln x}) + e^{\sin^{-1}x \ln(\cos^{-1}y)} = x^2$$

$$\begin{aligned}
 & e^{y \ln x} \left(\frac{y}{x} + \ln xy \right) \\
 \Rightarrow & \frac{x}{\sqrt{1 - (e^{y \ln x})^2}} \\
 & + e^{\sin^{-1} x \ln(\cos^{-1} y)} \left[\sin^{-1} x \frac{1}{\cos^{-1} y} * \frac{-y'}{\sqrt{1 - y^2}} \right. \\
 & \left. + \ln(\cos^{-1} y) \frac{1}{\sqrt{1 - x^2}} \right] = 2x
 \end{aligned}$$

D) Evaluate the following integrals:-

- 1) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} \frac{x}{2} + c$

$f = x$
note: $[df = dx$
 $a = 2$
- 2) $\int \frac{dx}{x\sqrt{1-\ln^2 x}} = \sin^{-1}(\ln x) + c$

$f = \ln x$
note: $[df = \frac{1}{x} dx$
 $a = 1$
- 3) $\int \frac{x dx}{\sqrt{1-x^2}} = -\frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} * -2x dx = -\frac{1}{2} \frac{\sqrt{1-x^2}}{1/2} = -\sqrt{1-x^2} + c$
- 4) $\int \frac{\sin^{-1} x dx}{\sqrt{1-x^2}} = \frac{(\sin^{-1})^2}{2} + c$
- 5) $\int \frac{2^x dx}{1+4^x} = \int \frac{2^x dx}{1+(2^x)^2} = \frac{1}{\ln 2} \tan^{-1} 2^x + c$

Note: $[f = 2^x$
 $df = 2^x \ln 2 dx$
 $a = 1$
- 6) $\int \frac{x^2 dx}{\sqrt{1-x^6}} = \frac{1}{3} \int \frac{3x^2 dx}{\sqrt{1-(x^3)^2}} = \frac{1}{3} \sin^{-1}(x^3) + c$

Note: $[f = x^3$
 $df = 3x^2 dx$
 $a = 1$
- 7) $\int \frac{dx}{3+2x^2} = \frac{1}{2} \int \frac{dx}{\frac{3}{2}+x^2} = \frac{1}{2} \frac{1}{\sqrt{\frac{3}{2}}} \tan^{-1} \left(\frac{x}{\sqrt{\frac{3}{2}}} \right) + c$

note: $f = x$
 $df = dx$
 $[a = \sqrt{\frac{3}{2}}$
- 8) $\int \frac{(1+x) dx}{4+x^2} = \int \frac{dx}{4+x^2} + \frac{1}{2} \int \frac{2x dx}{4+x^2}$

$$= \frac{1}{2} \tan^{-1} \frac{x}{2} + \frac{1}{2} \ln|4 + x^2| + C$$

$$9) \int \frac{dx}{x^2+4x+8} = \int \frac{dx}{x^2+4x+4-4+8} = \int \frac{dx}{(x+2)^2+4} \quad \text{note: } \begin{cases} f = x + 2 \\ df = dx \\ a = 2 \end{cases}$$

$$= \frac{1}{2} \tan^{-1} \left(\frac{x+2}{2} \right) + c$$

$$10) \int \frac{dx}{\sqrt{6x-x^2-8}} = \int \frac{dx}{\sqrt{-(x^2-6x+8)}} = \int \frac{dx}{\sqrt{-(x^2-6x+9-9+8)}}$$

$$= \int \frac{dx}{\sqrt{-(x-3)^2-1}} = \int \frac{dx}{\sqrt{1-(x-3)^2}} \quad \text{note: } \begin{cases} f = x - 3 \\ df = dx \\ a = 1 \end{cases}$$

$$= \sin^{-1}(x-3) + c$$

$$11) \int \frac{\cos x \, dx}{\sqrt{1-\sin^2 x}} = \sin^{-1}(\sin x) = x + c$$

$$12) \int \frac{(\tan^{-1} \sqrt{x})^3}{\sqrt{x}(1+x)} = 2 \int \frac{(\tan^{-1} \sqrt{x})^3}{2\sqrt{x}(1+x)} = \frac{2(\tan^{-1} \sqrt{x})^4}{4} + c$$

$$13) \int \frac{3x^3-4x^2+3}{x^2+1} dx \quad \text{ملاحظة: إذا كانت الدالة كسرية واس البسط اكبر او يساوي المقام نقوم بتقسيم البسط على المقام أولاً.}$$

$$= \int (3x - 4) + \frac{4}{x^2 + 1} dx$$

$$= \int 3x \, dx - 4 \int dx + 4 \int \frac{1}{x^2 + 1} dx$$

$$= \frac{3x^2}{2} - 4x + 4 \tan^{-1}(x) + c$$

$$\begin{array}{r} 3x - 4 \\ x^2 + 1 \overline{) 3x^3 - 4x^2 + 3x} \\ \underline{3x^3 + 3x} \\ -4x^2 - 4 \\ \underline{-4x^2 - 4} \\ +4 \end{array}$$