**Al- Furat Al- Awsat Technical University**
**Najaf Technical Institute**
**Medical device Department**

**Stage:** First Stage
**Academic Year:** 2025-2026
**Academic System:** Annual System – 30 Weeks
**Total Contact Hours:** 120 Hours
**Total Units:** 8 Units
**Weekly Hours:** 4 Hours
**Attendance:** Classroom Lectures + Practical
**Course Coordinator:** Asst. Prof. Dr. Mohammed Kareem Khashan
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# 1. COURSE DESCRIPTION

Aircraft Configurations is a first-stage course that introduces students to the principal structural units and configurations of fixed-wing aircraft.

The course covers the historical development of aircraft structures, major airframe units, structural stresses, wing configurations and construction, primary flight-control surfaces, dual-purpose flight-control surfaces, secondary and auxiliary flight surfaces, and landing-gear configurations.

The course combines theoretical knowledge with practical observation and identification of aircraft structural components and configurations. Students are expected to develop the ability to recognize major aircraft structural units, explain their functions, distinguish common configurations, and understand the relationship between aircraft structure, aerodynamic control, and structural loading.

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# 2. COURSE AIMS AND OBJECTIVES

The course aims to:

1. Introduce students to the historical development of aircraft structures.
2. Explain the principal structural units of fixed-wing aircraft.
3. Identify the main functions of the fuselage, wings, stabilizers, flight-control surfaces, and landing gear.
4. Introduce students to major structural stresses.
5. Explain tension, compression, torsion, shear, and bending.
6. Introduce common wing configurations and basic wing structural arrangements.
7. Explain the purpose and operation of primary flight-control surfaces.
8. Identify ailerons, elevators, stabilizers, and rudders.
9. Introduce dual-purpose flight-control surfaces.
10. Explain secondary and auxiliary flight surfaces.

11. Identify flaps, trim tabs, balance tabs, anti-balance tabs, servo tabs, spoilers, slats, slots, and leading-edge flaps.
12. Introduce common landing-gear configurations.
13. Explain fixed, tail-wheel, and tricycle landing-gear arrangements.
14. Develop students' practical aircraft-component identification skills.
15. Develop appropriate technical terminology related to aircraft structures and configurations.

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# 3. COURSE LEARNING OUTCOMES

At the end of the course, students should be able to:

1. Describe the historical development of aircraft structures.
2. Identify the principal units of a fixed-wing aircraft.
3. Explain the basic function of the fuselage.
4. Explain the function and structural importance of wings.
5. Identify stabilizers and explain their basic function.
6. Identify major flight-control surfaces.
7. Identify different landing-gear arrangements.
8. Explain the basic concepts of tension and compression.
9. Explain torsional and shear stresses.
10. Explain bending stress in aircraft structures.
11. Recognize common wing configurations.
12. Describe the basic structural arrangement of a wing.
13. Explain fundamental principles of wing construction.
14. Identify primary flight-control surfaces.
15. Explain the function of ailerons, elevators, stabilizers, and rudders.
16. Explain the concept of dual-purpose flight-control surfaces.
17. Identify secondary and auxiliary flight surfaces.
18. Explain the functions of flaps, tabs, spoilers, slats, and slots.
19. Distinguish between tail-wheel and tricycle landing gear.
20. Use appropriate technical English terminology related to aircraft configurations.

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# 4. TEACHING AND LEARNING STRATEGIES

The course may employ the following teaching and learning methods:

- Scientific and technical lectures.
- Practical demonstrations.
- Aircraft diagrams and technical illustrations.
- Component identification exercises.
- Classroom discussion.
- Guided questioning.
- Short diagnostic and achievement tests.
- Practical laboratory activities.
- Group activities.
- Aircraft configuration comparison.
- Technical assignments.
- Short scientific reports.
- Review sessions.
- Student presentations where appropriate.

# 5. ASSESSMENT METHODS

Assessment may include:

1. Weekly pre-tests.
2. Weekly post-tests.
3. Classroom participation.
4. Practical identification activities.
5. Homework assignments.
6. Short quizzes.
7. Practical examinations.
8. Periodic examinations.
9. Scientific reports.
10. Final examination.

# 6. OFFICIAL 30-WEEK COURSE SYLLABUS

| Weeks | Main Topic |
|---|---|
| **1–2** | History of Aircraft Structures |
| **3–7** | Principal Units for the Airframe of a Fixed-Wing Aircraft: Fuselage, Wings, Stabilizer, Flight-Control Surfaces, Landing Gear |
| **8–10** | Major Structural Stress: Tension, Compression, Torsion, Shear, Bending |
| **11–15** | Wings Configuration, Wing Structure, and Fundamental Design of Wing Construction |
| **16–19** | Flight-Control Surfaces: Primary Flight-Control Surface, Aileron, Elevator, Stabilizer, Rudder |
| **20–22** | Dual-Purpose Flight-Control Surfaces |
| **23–26** | Secondary/Auxiliary Flight Surfaces: Flaps, Trim Tabs, Balance Tabs, Anti-Balance Tabs, Servo Tabs, Spoilers, Slats, Slots, Leading-Edge Flap |
| **27–30** | Landing Gear: Fixed, Configuration of Airplane Landing Gear, Tail-Wheel Gear, Tricycle |

# 7. DETAILED WEEKLY COURSE MATERIAL

# WEEK 1

## History of Aircraft Structures – Early Development

### Learning Objectives

Students should be able to:

- Define the concept of aircraft structure.
- Explain why aircraft structures are important.
- Describe the early development of aircraft structures.
- Identify the basic relationship between aircraft weight, strength, and flight.

### Pre-Test

1. What is meant by an aircraft structure?
2. Why must an aircraft structure be strong?
3. What are the main requirements of an aircraft structure?

## Lecture Explanation

Aircraft structure refers to the physical framework that supports the aircraft and its components. The structure must carry the aircraft's weight and withstand forces produced during flight, landing, and other operating conditions.

The development of aircraft structures progressed from relatively simple constructions to more advanced structural arrangements. Early aircraft commonly used lightweight materials and simple structural frameworks.

Aircraft designers have always had to balance three important requirements:

- Strength
- Low weight
- Functional configuration

A structure that is excessively heavy reduces the efficiency of the aircraft, while a structure that is insufficiently strong cannot safely withstand operational loads.

## Key Terms

- Aircraft Structure
- Airframe
- Strength
- Weight
- Load
- Framework

## Practical Activity

Students examine aircraft structural diagrams and identify the main structural areas.

## Post-Test

1. Define aircraft structure.
2. State two important requirements of an aircraft structure.
3. Why is low structural weight important?

### Homework

Write a short report about the early development of aircraft structures.

### Learning Outcome

Students understand the basic concept and importance of aircraft structures.

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# WEEK 2

## History of Aircraft Structures – Development of Modern Airframes

### Learning Objectives

Students should be able to:

- Explain the development of aircraft airframes.
- Identify the importance of structural materials.
- Explain the transition toward more efficient structural designs.

### Pre-Test

1. What is an airframe?
2. Why are materials important in aircraft construction?
3. What is the relationship between strength and weight?

### Lecture Explanation

The development of aircraft structures has been closely related to improvements in materials, manufacturing techniques, and

aircraft design.

As aircraft developed, structural designers required materials and configurations capable of providing adequate strength while keeping structural weight as low as practical.

The airframe provides the basic structural arrangement of the aircraft and supports major units such as the fuselage, wings, stabilizers, control surfaces, and landing gear.

Modern aircraft structures therefore reflect the need for structural strength, efficient weight distribution, and functional integration of aircraft components.

## Key Terms

- Airframe
- Structural Material
- Structural Strength
- Weight Distribution
- Aircraft Configuration

## Practical Activity

Identify the principal structural units on an aircraft diagram.

## Post-Test

1. What is an airframe?
2. Why is structural weight important?
3. Name three major aircraft structural units.

## Homework

Prepare a labelled diagram showing the principal units of a fixed-wing aircraft.

## Learning Outcome

Students understand the development and general arrangement of aircraft airframes.

# WEEK 3

## Principal Airframe Units – Fuselage

### Learning Objectives

Students should be able to:

- Identify the fuselage.
- Explain its basic function.
- Identify the position of the fuselage within the airframe.

### Pre-Test

1. What is the fuselage?
2. Where is the fuselage located?
3. What is one major function of the fuselage?

### Lecture Explanation

The fuselage is the main body of a fixed-wing aircraft. It forms the central structural unit and provides the basic connection between major aircraft components.

Depending on the aircraft configuration, the fuselage may contain the cockpit, passenger or cargo areas, equipment, systems, and other components.

The fuselage also provides structural attachment points for components such as wings, stabilizers, and landing gear.

### Key Terms

- Fuselage
- Cockpit
- Airframe

- Structural Attachment
- Aircraft Body

### Practical Activity

Students identify the fuselage on several aircraft illustrations.

### Post-Test

1. Define fuselage.
2. State two functions of the fuselage.
3. Which major aircraft components may be attached to the fuselage?

### Homework

Draw and label the fuselage of a fixed-wing aircraft.

### Learning Outcome

Students can identify the fuselage and explain its basic structural role.

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# WEEK 4

## Principal Airframe Units – Wings

### Learning Objectives

Students should be able to:

- Identify aircraft wings.
- Explain their basic function.
- Recognize wings as major structural units.

## Pre-Test

1. What is a wing?
2. Where are wings normally located?
3. What is the main aerodynamic purpose of a wing?

## Lecture Explanation

Wings are major aircraft structural units. In a fixed-wing aircraft, the wings provide the primary lifting surfaces.

The wing must have sufficient structural strength to withstand aerodynamic and other operational loads.

Wing configuration may vary according to aircraft design requirements.

## Key Terms

- Wing
- Lifting Surface
- Wing Structure
- Aerodynamic Load
- Wing Configuration

## Practical Activity

Identify different wing arrangements from aircraft illustrations.

## Post-Test

1. What is the main function of a wing?
2. Why must wings be structurally strong?
3. What is meant by wing configuration?

## Homework

Draw a basic aircraft wing and label its main structural area.

### Learning Outcome

Students understand the structural and functional importance of aircraft wings.

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# WEEK 5

## Principal Airframe Units – Stabilizer

### Learning Objectives

Students should be able to:

- Identify stabilizers.
- Explain their basic function.
- Distinguish stabilizing surfaces from other aircraft components.

### Pre-Test

1. What is a stabilizer?
2. Where are stabilizers located?
3. Why does an aircraft require stabilizing surfaces?

### Lecture Explanation

Stabilizers are aircraft surfaces associated with maintaining the aircraft's desired attitude and stability.

The horizontal stabilizer is associated with longitudinal stability, while the vertical stabilizer is associated with directional stability.

Stabilizers form an important part of the aircraft tail arrangement.

### Key Terms

- Stabilizer
- Horizontal Stabilizer
- Vertical Stabilizer
- Longitudinal Stability
- Directional Stability

### Practical Activity

Identify horizontal and vertical stabilizers on aircraft diagrams.

### Post-Test

1. What is a stabilizer?
2. What is the function of a horizontal stabilizer?
3. What is the function of a vertical stabilizer?

### Homework

Prepare a labelled drawing of an aircraft tail assembly.

### Learning Outcome

Students can identify the major stabilizing surfaces.

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# WEEK 6

## Principal Airframe Units – Flight-Control Surfaces

### Learning Objectives

Students should be able to:

- Define flight-control surfaces.

- Identify major control surfaces.
- Explain their basic purpose.

## Pre-Test

1. What is a flight-control surface?
2. Name one primary flight-control surface.
3. Why are control surfaces necessary?

## Lecture Explanation

Flight-control surfaces are movable aircraft surfaces used to influence aircraft motion and attitude.

Primary control surfaces include:

- Ailerons
- Elevator
- Rudder

Other aircraft surfaces provide additional aerodynamic functions.

## Key Terms

- Flight-Control Surface
- Aileron
- Elevator
- Rudder
- Control

## Practical Activity

Identify control surfaces on aircraft drawings.

## Post-Test

1. Define a flight-control surface.

2. Name three primary control surfaces.
3. What is the purpose of movable control surfaces?

### Homework

Prepare a table showing the name and basic function of the primary control surfaces.

### Learning Outcome

Students recognize the major flight-control surfaces.

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# WEEK 7

## Principal Airframe Units – Landing Gear

### Learning Objectives

Students should be able to:

- Identify landing gear.
- Explain its basic function.
- Recognize different landing-gear arrangements.

### Pre-Test

1. What is landing gear?
2. What is its main function?
3. Name one type of landing gear configuration.

### Lecture Explanation

Landing gear is the aircraft structural system used to support the aircraft during ground operations.

It allows the aircraft to:

- Stand on the ground.
- Taxi.
- Take off.
- Land.

Different aircraft may use different landing-gear configurations depending on their design.

## Key Terms

- Landing Gear
- Fixed Gear
- Tail-Wheel Gear
- Tricycle Gear
- Ground Operation

## Practical Activity

Identify landing-gear configurations from aircraft images.

## Post-Test

1. What is the purpose of landing gear?
2. Name two landing-gear configurations.
3. When does landing gear support the aircraft?

## Homework

Compare tail-wheel and tricycle landing gear.

## Learning Outcome

Students understand the basic purpose and configurations of aircraft landing gear.

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# WEEK 8

## Major Structural Stress – Tension and Compression

### Learning Objectives

Students should be able to:

- Define tension.
- Define compression.
- Identify basic structural loading conditions.

### Pre-Test

1. What is stress?
2. What is tension?
3. What is compression?

### Lecture Explanation

Aircraft structures are subjected to different types of loads and stresses.

**Tension** occurs when a structural member is subjected to forces that tend to pull it apart.

**Compression** occurs when forces tend to push or shorten a structural member.

A simplified relationship for normal stress is:

σ=FA\sigma = \frac{F}{A}

where:

- σ\sigma = normal stress
- FF = applied force
- AA = cross-sectional area

### Key Terms

- Stress
- Force
- Area
- Tension
- Compression

### Practical Activity

Use simple structural members or diagrams to demonstrate pulling and pushing loads.

### Post-Test

1. Define tension.
2. Define compression.
3. Write the basic normal-stress equation.

### Homework

Give two examples of structural members that may experience tension or compression.

### Learning Outcome

Students understand tension and compression as basic structural stresses.

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# WEEK 9

## Major Structural Stress – Torsion and Shear

### Learning Objectives

Students should be able to:

- Define torsion.
- Define shear.
- Distinguish torsion from shear.

## Pre-Test

1. What is torsion?
2. What is shear?
3. What type of loading causes twisting?

## Lecture Explanation

**Torsion** is associated with twisting of a structural member due to an applied torque.

**Shear** occurs when forces act parallel to a surface or section and tend to cause adjacent parts of a structure to slide relative to one another.

Aircraft structures can experience both torsional and shear loading during operation.

## Key Terms

- Torsion
- Torque
- Shear
- Shear Force
- Twisting

## Practical Activity

Demonstrate twisting using a simple rod and demonstrate shear using layered material.

## Post-Test

1. Define torsion.
2. Define shear.
3. What is torque associated with?

### Homework

Prepare a short comparison between torsion and shear.

### Learning Outcome

Students can distinguish torsional and shear loading.

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# WEEK 10

## Major Structural Stress – Bending and Stress Review

### Learning Objectives

Students should be able to:

- Define bending.
- Identify the major structural stresses.
- Review the relationship between loads and structural response.

### Pre-Test

1. What is bending?
2. Name three structural stresses.
3. What can cause bending in a structural member?

### Lecture Explanation

Bending occurs when a structural member is subjected to loads that cause it to curve.

Aircraft structures may experience combinations of:

- Tension

- Compression
- Torsion
- Shear
- Bending

These stresses may occur simultaneously in aircraft components.

### Key Terms

- Bending
- Structural Load
- Stress
- Deformation

### Practical Activity

Use a simple beam to demonstrate bending under an applied load.

### Post-Test

1. Define bending.
2. List the five major stresses covered in this unit.
3. Can more than one stress occur at the same time?

### Homework

Prepare a summary table of the five major structural stresses.

### Learning Outcome

Students understand the five major structural stresses introduced in the course.

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# WEEK 11

# Wing Configurations

## Learning Objectives

Students should be able to:

- Define wing configuration.
- Identify common wing arrangements.
- Compare basic wing positions.

## Pre-Test

1. What is wing configuration?
2. Why can aircraft wings have different configurations?
3. What is meant by wing position?

## Lecture Explanation

Wing configuration describes the general arrangement and geometry of the aircraft wing.

Aircraft may use different wing arrangements depending on the intended design and operating requirements.

Students should learn to recognize wing configurations from technical drawings and aircraft illustrations.

## Key Terms

- Wing Configuration
- Wing Position
- Planform
- Wing Arrangement

## Practical Activity

Identify different wing configurations from aircraft diagrams.

## Post-Test

1. Define wing configuration.
2. Why are different wing configurations used?
3. Identify one wing characteristic that may vary.

### Homework

Collect examples of different aircraft wing configurations.

### Learning Outcome

Students recognize basic wing configurations.

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# WEEK 12

## Wing Structure – Basic Structural Arrangement

### Learning Objectives

Students should be able to:

- Identify the basic parts of wing structure.
- Explain the role of structural members.
- Understand the relationship between wing structure and loads.

### Pre-Test

1. What is wing structure?
2. Why does a wing require internal structural members?
3. What type of loads can affect a wing?

### Lecture Explanation

The wing structure must support aerodynamic loads and transfer these loads through the aircraft structure.

A wing may include structural members arranged to provide the necessary strength and stiffness.

The detailed arrangement depends on aircraft design.

### Key Terms

- Wing Structure
- Structural Member
- Load
- Stiffness
- Strength

### Practical Activity

Identify major structural regions on a wing diagram.

### Post-Test

1. Why does a wing require internal structure?
2. What is the purpose of structural members?
3. What types of loads can act on a wing?

### Homework

Draw a simplified wing structural arrangement.

### Learning Outcome

Students understand the basic structural organization of wings.

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# WEEK 13

## Fundamental Design of Wing Construction

## Learning Objectives

Students should be able to:

- Explain fundamental principles of wing construction.
- Relate structural design to loading.
- Identify major construction concepts.

## Pre-Test

1. What is wing construction?
2. Why must a wing be strong?
3. What is the relationship between structure and load?

## Lecture Explanation

Wing construction is designed to provide the required strength while maintaining an appropriate structural arrangement.

The design must consider the loads applied to the wing during aircraft operation.

The internal structure distributes loads through the wing and toward the main aircraft structure.

## Key Terms

- Wing Construction
- Structural Design
- Load Distribution
- Strength
- Stiffness

## Practical Activity

Examine a simplified wing cross-section and identify structural regions.

## Post-Test

1. Why is wing structural design important?
2. What is load distribution?
3. What should a wing structure provide?

### Homework

Write a short explanation of fundamental wing construction.

### Learning Outcome

Students understand the relationship between wing construction and structural loading.

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# WEEK 14

## Wing Structural Design – Loads and Arrangement

### Learning Objectives

Students should be able to:

- Explain how loads influence wing structure.
- Identify the importance of structural arrangement.
- Discuss the relationship between design and strength.

### Pre-Test

1. What is a structural load?
2. Why should loads be considered during wing design?
3. What is structural arrangement?

### Lecture Explanation

Aircraft wing design must account for loads acting on the wing. Structural members are arranged so that loads can be

transmitted through the wing and into the aircraft structure.

An appropriate arrangement helps the wing maintain its required shape and structural integrity.

### Key Terms

- Load Path
- Structural Arrangement
- Wing Strength
- Structural Integrity

### Practical Activity

Students trace a simplified load path through a wing diagram.

### Post-Test

1. Why are load paths important?
2. What is structural integrity?
3. How does structural arrangement affect wing strength?

### Homework

Draw and explain a simple wing load path.

### Learning Outcome

Students understand the relationship between wing loads and structural arrangement.

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# WEEK 15

## Wing Configuration and Structure – Review

## Learning Objectives

Students should be able to:

- Review wing configurations.
- Identify major structural concepts.
- Connect wing configuration with structural requirements.

## Pre-Test

1. What is wing configuration?
2. What is wing structure?
3. Why must wing structure withstand loads?

## Lecture Explanation

This week consolidates the material covered in Weeks 11–14.

Students review wing configuration, wing structure, construction, structural arrangement, and loading.

## Practical Activity

Group identification exercise using several aircraft diagrams.

## Post-Test

1. Define wing configuration.
2. Name two important wing structural concepts.
3. Explain why wing structures must withstand loads.

## Homework

Prepare a two-page review of wing configuration and construction.

## Learning Outcome

Students consolidate their understanding of wing configuration and structure.

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# WEEK 16

## Primary Flight-Control Surfaces – Overview

### Learning Objectives

Students should be able to:

- Define primary flight-control surfaces.
- Identify the main primary controls.
- Explain their general purpose.

### Pre-Test

1. What are primary flight-control surfaces?
2. Name the three main primary controls.
3. Why are flight controls necessary?

### Lecture Explanation

Primary flight-control surfaces are used to control the aircraft's principal rotational movements.

The main primary flight-control surfaces are:

- Aileron
- Elevator
- Rudder

Each surface performs a specific control function.

### Key Terms

- Primary Flight Control
- Aileron
- Elevator
- Rudder
- Aircraft Motion

### Practical Activity

Label the three primary flight-control surfaces on an aircraft diagram.

### Post-Test

1. Name the primary flight-control surfaces.
2. What is the function of an aileron?
3. What is the function of a rudder?

### Homework

Prepare a table of primary flight-control surfaces and their functions.

### Learning Outcome

Students identify the main primary flight-control surfaces.

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# WEEK 17

## Aileron

### Learning Objectives

Students should be able to:

- Identify the aileron.

- Explain its basic function.
- Relate aileron movement to aircraft rolling motion.

## Pre-Test

1. What is an aileron?
2. Where is the aileron generally located?
3. What aircraft motion is associated with aileron control?

## Lecture Explanation

The aileron is a primary flight-control surface associated with control of aircraft roll.

Ailerons are generally located on the wings and operate through differential movement.

When one aileron moves in one direction, the opposite aileron moves in the opposite direction, producing a difference in aerodynamic force and influencing aircraft roll.

## Key Terms

- Aileron
- Roll
- Wing
- Differential Movement

## Practical Activity

Use a model or diagram to demonstrate aileron movement.

## Post-Test

1. What is an aileron?
2. What aircraft motion does it control?
3. Where is it located?

## Homework

Draw an aircraft and clearly label both ailerons.

### Learning Outcome

Students understand the basic purpose and location of the aileron.

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# WEEK 18

## Elevator and Stabilizer

### Learning Objectives

Students should be able to:

- Identify the elevator.
- Identify the horizontal stabilizer.
- Explain their basic relationship.

### Pre-Test

1. What is an elevator?
2. What is a horizontal stabilizer?
3. Where are they located?

### Lecture Explanation

The elevator is a primary flight-control surface associated with control of aircraft pitch.

The horizontal stabilizer forms part of the aircraft tail arrangement and contributes to longitudinal stability.

The elevator and horizontal stabilizer work together as important components of the horizontal tail assembly.

### Key Terms

- Elevator
- Horizontal Stabilizer
- Pitch
- Longitudinal Stability
- Tail Assembly

### Practical Activity

Identify elevator and horizontal stabilizer on aircraft diagrams.

### Post-Test

1. What is the function of the elevator?
2. What is the function of the horizontal stabilizer?
3. What aircraft motion is associated with elevator control?

### Homework

Draw the horizontal tail and label the elevator and stabilizer.

### Learning Outcome

Students distinguish the elevator from the horizontal stabilizer.

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# WEEK 19

## Rudder and Primary Control Review

### Learning Objectives

Students should be able to:

- Identify the rudder.

- Explain its basic function.
- Review all primary flight-control surfaces.

## Pre-Test

1. What is a rudder?
2. What aircraft motion does it control?
3. Name the three primary controls.

## Lecture Explanation

The rudder is a primary flight-control surface associated with directional control of the aircraft.

It is generally mounted on the vertical tail assembly.

The three primary control surfaces studied are:

| Control Surface | Basic Control |
|---|---|
| Aileron | Roll |
| Elevator | Pitch |
| Rudder | Yaw |

## Practical Activity

Students complete a control-surface identification worksheet.

## Post-Test

1. What does the rudder control?
2. Where is the rudder located?
3. Complete the aileron–elevator–rudder control table.

## Homework

Prepare a labelled diagram showing all three primary control surfaces.

### Learning Outcome

Students can identify and distinguish the three primary flight-control surfaces.

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# WEEK 20

## Dual-Purpose Flight-Control Surfaces – Introduction

### Learning Objectives

Students should be able to:

- Define dual-purpose flight-control surfaces.
- Explain why a control surface may perform more than one function.
- Recognize the concept in aircraft configurations.

### Pre-Test

1. What is a flight-control surface?
2. Can one surface perform more than one function?
3. Why might aircraft designers use multifunctional surfaces?

### Lecture Explanation

A dual-purpose flight-control surface is a surface designed to contribute to more than one aircraft control or aerodynamic function.

The specific arrangement depends on aircraft design.

The use of multifunctional surfaces can influence the aircraft configuration and control system.

### Key Terms

- Dual-Purpose
- Flight Control
- Multifunctional Surface
- Aircraft Configuration

### Practical Activity

Students examine aircraft diagrams and identify surfaces with multiple functions.

### Post-Test

1. What is a dual-purpose control surface?
2. Why can a surface have more than one function?
3. What determines the specific function of a control surface?

### Homework

Find an aircraft example that uses a multifunctional control-surface arrangement.

### Learning Outcome

Students understand the concept of dual-purpose flight-control surfaces.

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# WEEK 21

## Dual-Purpose Flight-Control Surfaces – Configuration

### Learning Objectives

Students should be able to:

- Identify dual-purpose configurations.
- Explain their general purpose.

- Compare them with conventional control arrangements.

## Pre-Test

1. What is a dual-purpose surface?
2. What is meant by aircraft configuration?
3. Why can control surfaces be combined?

## Lecture Explanation

Some aircraft configurations combine control functions within a single surface or arrangement.

Such configurations must provide appropriate control authority while maintaining suitable structural and aerodynamic characteristics.

Students should focus on identifying the configuration and understanding its intended function.

## Practical Activity

Compare conventional and combined control-surface diagrams.

## Post-Test

1. What is a combined control surface?
2. What is the main advantage of combining functions?
3. What must be considered when designing such a surface?

## Homework

Prepare a comparison between conventional and dual-purpose control surfaces.

## Learning Outcome

Students can recognize basic dual-purpose configurations.

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# WEEK 22

## Dual-Purpose Flight-Control Surfaces – Review

### Learning Objectives

Students should be able to:

- Review the concept of dual-purpose controls.
- Identify examples from diagrams.
- Explain the purpose of combined control arrangements.

### Pre-Test

1. Define dual-purpose control.
2. Why are control functions sometimes combined?
3. What is the role of aircraft configuration?

### Lecture Explanation

This week reviews the concepts presented in Weeks 20 and 21.

Students should be able to identify multifunctional surfaces and explain why aircraft designers may use different control configurations.

### Practical Activity

Group classification of aircraft control-surface diagrams.

### Post-Test

1. Define a dual-purpose flight-control surface.
2. State one reason for using a combined control arrangement.
3. Identify a dual-purpose surface from a diagram.

### Homework

Prepare a short review report on dual-purpose flight-control surfaces.

### Learning Outcome

Students consolidate their understanding of dual-purpose flight-control configurations.

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# WEEK 23

## Secondary/Auxiliary Flight Surfaces – Flaps and Trim Tabs

### Learning Objectives

Students should be able to:

- Define secondary/auxiliary flight surfaces.
- Identify flaps.
- Identify trim tabs.
- Explain their general purposes.

### Pre-Test

1. What are secondary flight surfaces?
2. What is a flap?
3. What is a trim tab?

### Lecture Explanation

Secondary or auxiliary flight surfaces provide additional aerodynamic functions.

**Flaps** are movable surfaces associated with changing the aerodynamic characteristics of the wing.

**Trim tabs** are small movable surfaces used to assist in maintaining a desired control condition and reducing the control effort required.

### Key Terms

- Flap
- Trim Tab
- Auxiliary Surface
- Aerodynamic Function

### Practical Activity

Identify flaps and trim tabs on aircraft diagrams.

### Post-Test

1. What is a flap?
2. What is a trim tab?
3. Why are auxiliary flight surfaces used?

### Homework

Compare flaps and trim tabs.

### Learning Outcome

Students can identify and explain the general functions of flaps and trim tabs.

---

# WEEK 24

## Balance Tabs, Anti-Balance Tabs and Servo Tabs

### Learning Objectives

Students should be able to:

- Identify balance tabs.
- Identify anti-balance tabs.
- Identify servo tabs.
- Distinguish their general functions.

## Pre-Test

1. What is a balance tab?
2. What is an anti-balance tab?
3. What is a servo tab?

## Lecture Explanation

Tabs are small movable surfaces associated with control surfaces.

Their functions depend on the specific aircraft control-system arrangement.

A **balance tab** can assist in reducing control forces.

An **anti-balance tab** can be arranged to increase control feel or force characteristics.

A **servo tab** can assist in moving a larger control surface.

## Key Terms

- Balance Tab
- Anti-Balance Tab
- Servo Tab
- Control Force

## Practical Activity

Students identify the different tabs on technical diagrams.

### Post-Test

1. What is the purpose of a balance tab?
2. What is an anti-balance tab?
3. What is a servo tab?

### Homework

Prepare a comparison table for the three types of tabs.

### Learning Outcome

Students distinguish the basic functions of balance, anti-balance, and servo tabs.

---

# WEEK 25

## Spoilers, Slats and Slots

### Learning Objectives

Students should be able to:

- Identify spoilers.
- Identify slats.
- Identify slots.
- Explain their general aerodynamic purposes.

### Pre-Test

1. What is a spoiler?
2. What is a slat?
3. What is a slot?

## Lecture Explanation

**Spoilers** are surfaces used to modify the aerodynamic characteristics of a wing and can reduce lift while increasing drag when deployed.

**Slats** are leading-edge devices used to modify airflow over the wing.

A **slot** is an opening or passage that allows airflow to pass through or around a wing leading-edge arrangement and can help maintain suitable airflow conditions.

## Key Terms

- Spoiler
- Slat
- Slot
- Leading Edge
- Lift
- Drag

## Practical Activity

Identify spoilers, slats, and slots on aircraft diagrams.

## Post-Test

1. What is a spoiler?
2. What is a slat?
3. What is a slot?

## Homework

Prepare a comparison between spoilers, slats, and slots.

## Learning Outcome

Students can identify and describe the basic purposes of these auxiliary surfaces.

# WEEK 26

## Leading-Edge Flaps and Auxiliary-Surface Review

### Learning Objectives

Students should be able to:

- Identify leading-edge flaps.
- Explain their general purpose.
- Review auxiliary flight surfaces.

### Pre-Test

1. What is a leading-edge flap?
2. Where is it located?
3. Name three auxiliary flight surfaces.

### Lecture Explanation

Leading-edge flaps are movable aerodynamic surfaces located at or near the leading edge of a wing.

They can modify the aerodynamic characteristics of the wing.

This week also reviews:

- Flaps
- Trim tabs
- Balance tabs
- Anti-balance tabs
- Servo tabs
- Spoilers
- Slats

- Slots
- Leading-edge flaps

### Practical Activity

Students label all auxiliary flight surfaces on a wing diagram.

### Post-Test

1. What is a leading-edge flap?
2. Where is it located?
3. Name five auxiliary flight surfaces.

### Homework

Prepare a complete classification chart of secondary and auxiliary flight surfaces.

### Learning Outcome

Students understand the major secondary and auxiliary flight surfaces included in the syllabus.

---

# WEEK 27

## Landing Gear – Fixed Configuration

### Learning Objectives

Students should be able to:

- Define fixed landing gear.
- Identify fixed landing gear.
- Explain its basic arrangement.

## Pre-Test

1. What is landing gear?
2. What is fixed landing gear?
3. What is the purpose of landing gear?

## Lecture Explanation

Fixed landing gear remains in an extended position during flight rather than being retracted into the aircraft structure.

The configuration provides a simple structural arrangement for ground operations.

Students should recognize fixed landing gear from aircraft illustrations.

## Key Terms

- Fixed Landing Gear
- Landing Gear
- Ground Support
- Gear Leg
- Wheel

## Practical Activity

Identify fixed landing gear on different aircraft images.

## Post-Test

1. Define fixed landing gear.
2. Does fixed landing gear retract?
3. What is its basic function?

## Homework

Draw an aircraft with fixed landing gear and label its major components.

### Learning Outcome

Students recognize fixed landing-gear configurations.

---

# WEEK 28

## Aircraft Landing-Gear Configurations

### Learning Objectives

Students should be able to:

- Define landing-gear configuration.
- Identify common arrangements.
- Compare basic landing-gear layouts.

### Pre-Test

1. What is landing-gear configuration?
2. Name one landing-gear arrangement.
3. Why do aircraft use different landing-gear configurations?

### Lecture Explanation

Landing-gear configuration refers to the arrangement of the aircraft's landing-gear components.

Different arrangements are selected according to aircraft design requirements.

The course focuses particularly on:

- Fixed landing gear
- Tail-wheel landing gear
- Tricycle landing gear

### Practical Activity

Students classify aircraft diagrams according to landing-gear configuration.

### Post-Test

1. What is landing-gear configuration?
2. Name three configurations discussed in the course.
3. What determines the selection of a landing-gear arrangement?

### Homework

Prepare a comparison table of landing-gear configurations.

### Learning Outcome

Students can classify basic landing-gear configurations.

---

# WEEK 29

## Tail-Wheel Landing Gear

### Learning Objectives

Students should be able to:

- Identify tail-wheel landing gear.
- Explain its basic arrangement.
- Distinguish it from tricycle landing gear.

### Pre-Test

1. What is tail-wheel landing gear?

2. Where is the small wheel located?
3. What is the difference between tail-wheel and tricycle gear?

## Lecture Explanation

In a tail-wheel landing-gear configuration, the main landing gear supports the forward or central part of the aircraft while a smaller wheel or support is located toward the rear.

This configuration is also commonly described as conventional landing gear.

Students should identify the configuration from aircraft side-view diagrams.

## Key Terms

- Tail Wheel
- Main Gear
- Conventional Gear
- Landing Gear
- Aircraft Ground Attitude

## Practical Activity

Compare tail-wheel and tricycle aircraft diagrams.

## Post-Test

1. What is tail-wheel landing gear?
2. Where is the tail wheel located?
3. What is another name commonly associated with this configuration?

## Homework

Draw and label a tail-wheel landing-gear arrangement.

## Learning Outcome

Students can identify and describe tail-wheel landing gear.

---

# WEEK 30

## Tricycle Landing Gear and Final Course Review

### Learning Objectives

Students should be able to:

- Identify tricycle landing gear.
- Explain its basic arrangement.
- Compare tricycle and tail-wheel configurations.
- Review the complete course.

### Pre-Test

1. What is tricycle landing gear?
2. Where is the nose gear located?
3. How does tricycle gear differ from tail-wheel gear?

### Lecture Explanation

Tricycle landing gear consists of a main landing-gear arrangement together with a nose landing gear located toward the front of the aircraft.

The configuration differs from tail-wheel landing gear because the additional support is located at the nose rather than the rear.

### Comparison

| Feature | Tail-Wheel Gear | Tricycle Gear |
|---|---|---|
| Additional wheel/support | Rear | Front |

| Feature | Tail-Wheel Gear | Tricycle Gear |
|---|---|---|
| Nose gear | No | Yes |
| Tail wheel | Yes | No |
| Basic configuration | Main gear + tail wheel | Main gear + nose gear |

## Final Course Review

Students review:

1. History of aircraft structures.
2. Principal airframe units.
3. Fuselage.
4. Wings.
5. Stabilizers.
6. Flight-control surfaces.
7. Landing gear.
8. Structural stresses.
9. Wing configurations.
10. Wing construction.
11. Primary flight controls.
12. Dual-purpose controls.
13. Secondary and auxiliary surfaces.
14. Landing-gear configurations.

## Practical Activity

Final aircraft configuration identification exercise.

Students are given aircraft diagrams and asked to identify:

- Fuselage
- Wings
- Stabilizers
- Ailerons
- Elevator
- Rudder

- Auxiliary surfaces
- Landing gear

### Post-Test

1. What is tricycle landing gear?
2. What is the difference between tricycle and tail-wheel landing gear?
3. Name five major aircraft structural or configuration components studied in this course.

### Homework

Prepare a final course summary covering all 30 weeks.

### Learning Outcome

Students demonstrate an integrated understanding of aircraft structures and configurations.

---

# 8. COURSE-WIDE PRE-TEST

## Diagnostic Assessment

### Section A – Multiple Choice

1. The main body of a fixed-wing aircraft is called:
    - A. Wing
    - B. Fuselage
    - C. Rudder
    - D. Stabilizer
2. The primary control surface associated with roll is:
    - A. Elevator
    - B. Rudder
    - C. Aileron
    - D. Flap

3. The primary control surface associated with pitch is:
    - A. Elevator
    - B. Rudder
    - C. Spoiler
    - D. Aileron
4. The primary control surface associated with directional control is:
    - A. Flap
    - B. Rudder
    - C. Elevator
    - D. Slat
5. A force that tends to pull a structural member apart produces:
    - A. Compression
    - B. Tension
    - C. Torsion
    - D. Bending
6. A twisting structural load is associated with:
    - A. Torsion
    - B. Compression
    - C. Tension
    - D. Shear
7. A flap is generally considered:
    - A. An engine component
    - B. An auxiliary aerodynamic surface
    - C. Landing gear
    - D. A fuselage component
8. A landing gear configuration with a nose wheel is:
    - A. Tail-wheel
    - B. Tricycle
    - C. Fixed only
    - D. None

## Section B – Short Answer

1. Define an aircraft structure.
2. Name the main primary flight-control surfaces.
3. Name three landing-gear configurations.

4. List three structural stresses.
5. Define wing configuration.

# Aerospace Structures

## Chapter 1.
## Structural description of the aircraft
### Loads on structural components

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**



# Index

## CHAPTER 1. Structural description of the aircraft

## Loads on structural components


- Introduction
- Load types
- Load models
- Load factor
- References

# Introduction

## Structure definition

"The set of resistive elements capable of maintaining its forms and characteristics over time, under the action of external load“
(E. Torroja)

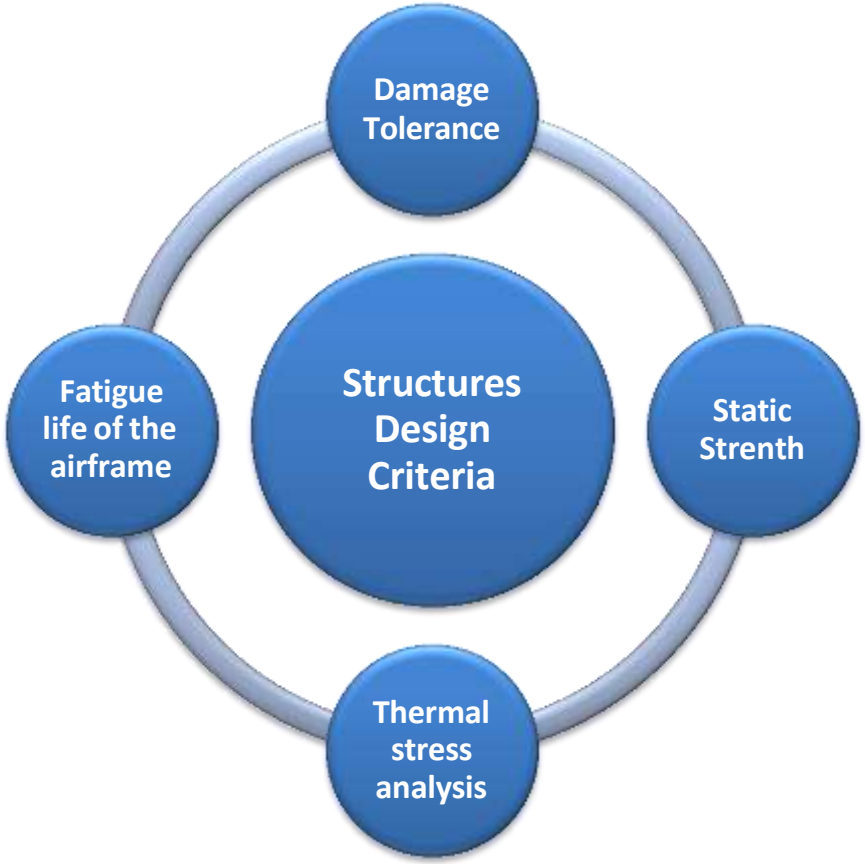
## Structure function

- Transmit and resist the applied loads
- Provide an aerodynamic shape
- Protect passenger and payload
- Support of aircraft systems

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Introduction

## ❑ Structural requirements




- ✓ **Maintainability**
- ✓ **Productivity**
- ✓ **Crashworthiness**

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

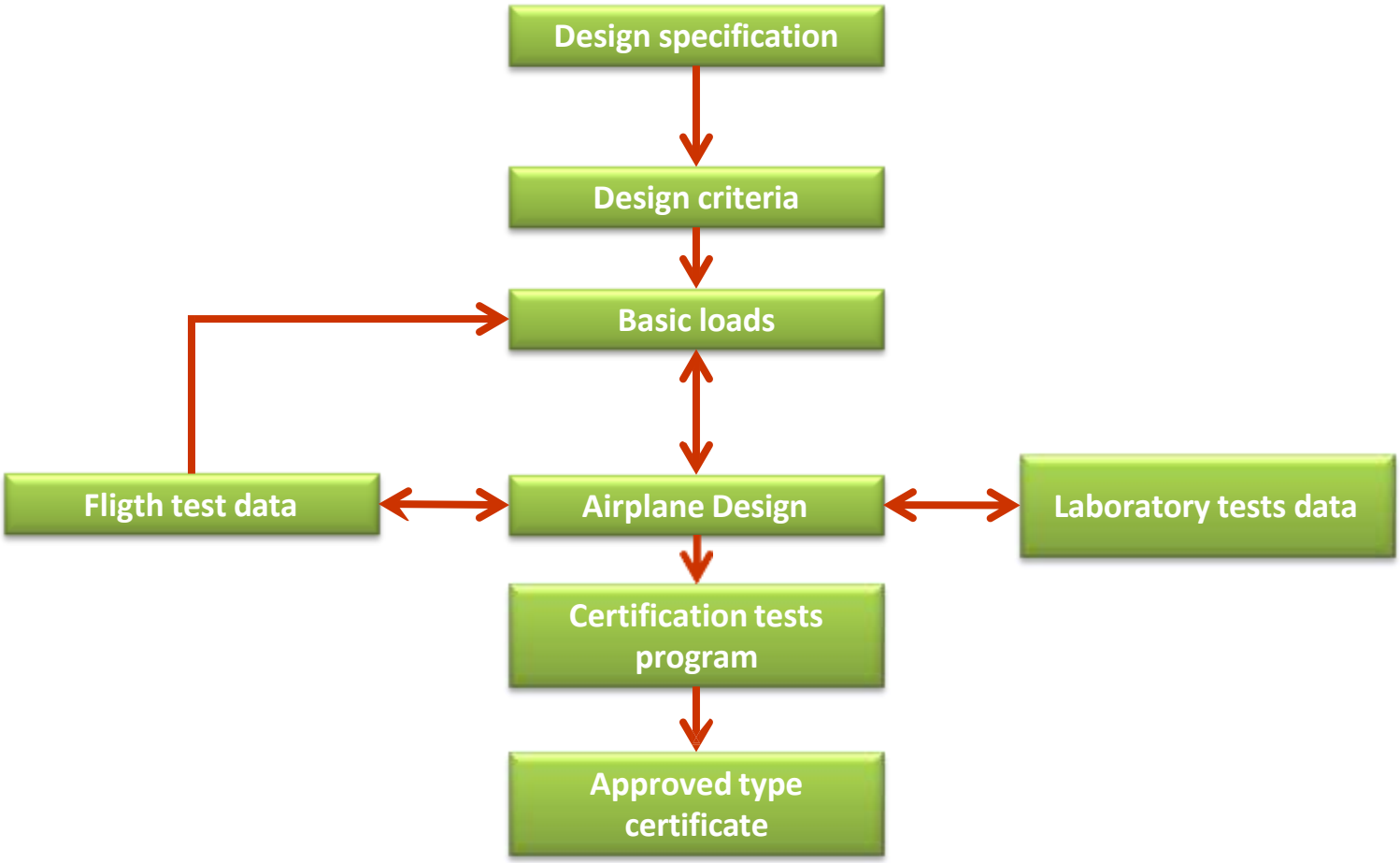
# Introduction

## ❑ Phases of structural design

- Specification of function an design criteria
- Determination of applied load
- Calculation of internal element loads
- Determination of allowable element strength
- Experimental test

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

## ❑ Phases of structural design

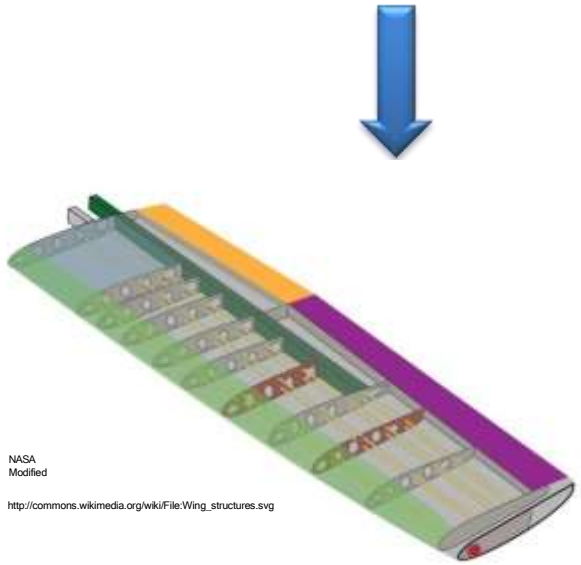



Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Introduction

## ❑ Types of structural analysis

Aircraft loads



NASA
Modified

http://commons.wikimedia.org/wiki/File:Wing_structures.svg

- Static analysis
- Damage tolerance
- Dynamics analysis/impact
- Aeroelastic analysis

# Introduction

## ❑ Static structural analysis

**Strength**

- Static ultimate strength
- Deformation of undamaged structure

Failure analysis 

**Buckling**

**Damage tolerance**

- Fatigue crack initiation
- Fatigue life of damaged structure
- Static residual strength of damaged structure

# Introduction

## ❑ Static structural analysis

| Mode of failure | Design criteria | Allowables data |
|---|---|---|
| static ultimate strength | Structure must support ultimate loads without failure for 3 seconds | Static properties |
| Deformation of undamaged structure | Deformation of the structure at limit loads may not interfere with safe operation of the element | Static properties and creep properties for elevated temperature conditions |
| Fatigue crack initiation of undamaged structures | -Fail-safe structure must meet customer service life requirements for operational loading conditions<br>-Safe life components must remain crack free in service. Replacements times must be specified for limited life components | -Crack growth properties<br>-Fracture toughness properties |
| Residual static strength of damaged structure | -Fail-safe structure must support 80-100% limit loads without catastrophic failure<br>-A single member failed in redundant structure or partial failure in monolithic structure | -Static properties<br>-Fracture toughness properties |
| Crack growth life of damaged structure | -For fail-safe structure inspection techniques and frequency must be specified to minimize risk of catastrophic failures<br>-For safe-life structures must define inspection techniques and frequencies and, replacement times so that probability of failure due to fatigue cracking is extremely remote | -Crack growth properties<br>-Fracture toughness properties |

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Load types

## ❑ Aircraft loads

The aircraft loads are those forces applied to the aircraft structural elements to establish the strength level of the airplane

- Flight loads
- Ground handling loads

The structural design is dependent on load therefore the loads must be determined early in the design

- Static loads
- Dynamics loads

- External loads
- Internal loads

# Load types

## ❑ Loads classification

**Flight Loads:**

- Symmetric maneuvers
- Asymmetric maneuvers
- Control deflection
- Gust loads

**Ground Handling:**

- Take off
- Landing
- Taxiing (asymmetric braking, turning etc.)
- Towing, Pivoting etc.
- Catapulted take-off
- Arrested landing
- Landing in water

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

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# Load types

## ❑ Loads classification

**Surface loads**

- Aerodynamic loads
- Hydrostatic pressure

**Body forces**

- Weight
- Inertial loads

**One-point loads**

# Load types

## ❑ Loads classification

**External Loads**

- Aerodynamic loads
- Take off/Landing loads
- Impact loads

**Local and Internal Loads**

- Max./min. aerodynamic pressures (outer surfaces)
- Local accelerations
- System pressures
- Bay pressures (pressurised areas)
- Hydrostatic pressures ( fuel tanks)
- Intake duct pressures
- Engine thrust
- Component interaction

# Load types

## ❑ Loads classification

**Static loads**

**Dynamic Loads:**

- Buffet (Outer wing, vertical fin buffet etc.)
- Dynamic Gust
- Vibrations
- Acoustic Noise
- Limit cycle oscillation
- Shimmy (Undercarriage)
- Engine hammershock conditions (Duct)

**Fatigue loads**

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Load models

First of all the STRUCTURAL DESIGN CRITERIA (SDC) are prepared as a basic for design, and then a LOADS MODEL (LM) is built. The LM is based on:

- SDC (Standards)
- Aerodynamic data (wind tunnel or flight tests)
- Flight mechanic data (FEMs)
- Weight data (Structural, systems and role equipment masses and mass distributions)
- Balance data

The load are analysed for all manoeuvres. From the load time history the external load distribution for any component (bending moment, torque, shear forces, etc) are determined for each component (fuselage, wing, etc).

# Load models

## ❑ Model types

Aerodynamic Panel model is considered just a geometric interface and can be used to deal with data of different nature:

- Theoretical coming from simplified potential flow tools.
- Theoretical, based upon more sophisticated CFD tools (Euler and/or Navier-Stokes codes)
- Experimental based upon W/T pressure plotting.
- Mix of previous, with an experimental basis supplemented with theoretical analyses for the areas non covered by tests.

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Load models

## ❑ Limit and ultimate loads

FAR Part 25. Sec. 25.301
Strength requirements are specified in terms of limit loads and ultimate loads

Unless otherwise provided, prescribed loads are limit loads.

Limit Loads are the maximum loads which the aircraft may encounter at any time during its lifetime. FAR Part 25 (and most other regulations) specifies that there be no permanent deformation of the structure at limit load.

Limit load cases are relevant for fatigue design requirements

Ultimate loads are load which may occur once in the lifetime of an aircraft. They are defined as the limit loads times a safety factor. The structure must be able to withstand the ultimate load for at least 3 seconds without failure. Permanent deformation may occur though.

Ultimate load cases are relevant for static strength requirements

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Load models

## ❑ Limit and ultimate loads

**Safety factor**

- Limitations of calculus model
- Variation in physical properties of materials
- Variation in fabrication standard
- Emergency conditions
- Variation of loads (gust random

| | |
|---|---|
| FAR Part 25 Section 25.303: | 1.50 |
| Research aircrafts: | 1.20 |
| Composite sailplanes: | 1.75 |

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# References


- **Aircraft Structures for engineering students**
  H.G. Megson
  Elsevier
  Section B1 Principles of stressed skin constuction
    Chapter 12 Structural components of aircraft
  Section 12 B2 Airworthisess and airframe loads
    Chapter 13 Airworthiness
    Chapter 14 Airframe loads

- **Airframe Structural Design**
  M.C.Y. Niu
  Hong Kong Conmilit Press LTD
  Section 3. Aircraft loads
    Chapter 8 Wing Box Structure
    Chapter 11Fuselage



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# Aerospace Structures

## Chapter 1.
## Structural description of the aircraft

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

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# CHAPTER 1. Structural description of the aircraft

## Structural components of aircraft


- Introduction
- Wing structure
- Fuselage structure
- Stabilizers structure
- References


Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Wing structure

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Wing structure

## Wing function




Oblivion8000 (talk).
http://commons.wikimedia.org/wiki/File:Airplane_Wing.jpg?uselang=es

The function of the wing is to produce lift

Main loads applied on the wing:

- ❑ Aerodynamic loads
- ❑ Weight

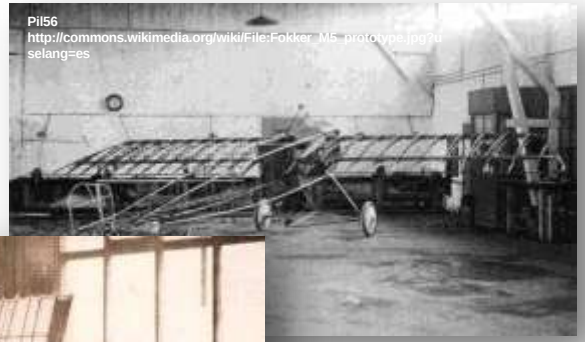
A wing works as a beam able to support the internal forces (Shear, bending and torsional moments)

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Wing structure

## Structures types

### Truss-type construction

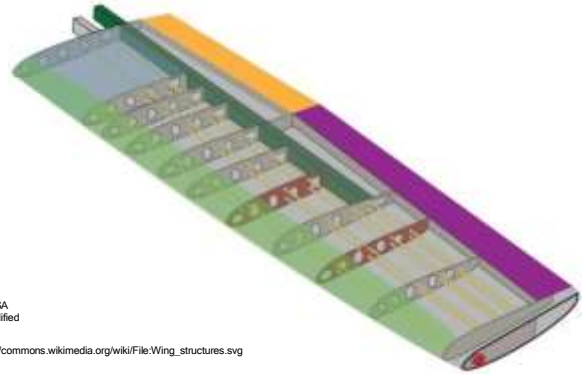


Pil56
http://commons.wikimedia.org/wiki/File:Fokker_M5_prototype.jpg?uselang=es



Seattle Municipal Archives
http://commons.wikimedia.org/wiki/File:Women_making_airplane_wings_c._1910s.jpg?uselang=es

### Stressed-skin construction



NASA
Modified

http://commons.wikimedia.org/wiki/File:Wing_structures.svg



russavia

http://commons.wikimedia.org/wiki/Category:BAE_Hawk_T1/T2_(Royal_Air_Force)?uselang=es#mediaviewer/File:Hawk_-_RIAT_2011_(6122004254).jpg

### Early wing structures



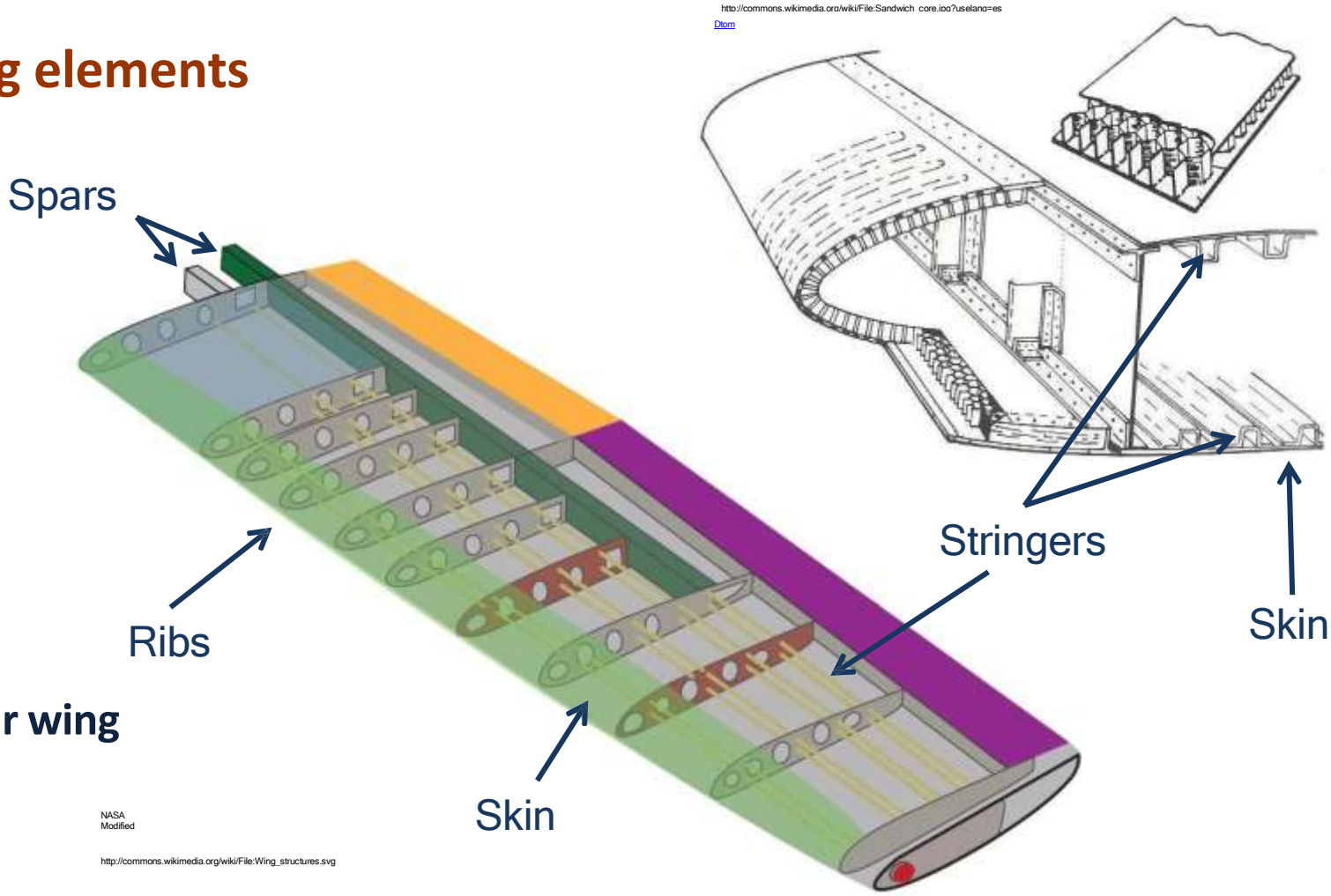
The PIPE

http://commons.wikimedia.org/wiki/File:Fokker_M5K-MG_E5-15.jpg?uselang=es

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Wing structure

- **Wing elements**

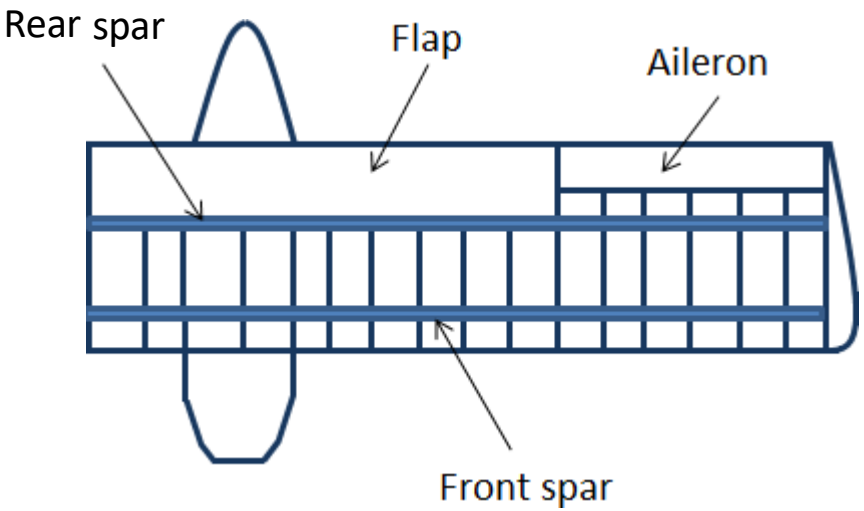



Principal structural elements of the wing

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Wing structure

- **Spars**




Torsion box structure: 2-3 spar

Front spar: 15-30% chord
Rear spar: 65-75% chord

## Semicantilever wings

Some aircraft have external struts for wing bracing

http://upload.wikimedia.org/wikipedia/commons/d/d7/Arctic_Sunwest_Twin_Otter.jpg?uselang=es








bus227
http://commons.wikimedia.org/wiki/File:FlyingWires.JPG



Credit: NASA Langley/Sean Smith
http://www.nasa.gov/centers/langley/multimedia/iotw-tdt-wing_prt.htm

Boeing Aerodynamic Efficiency Improvement Joined Wing




Boeing SUGAR Volt concept

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Wing structure

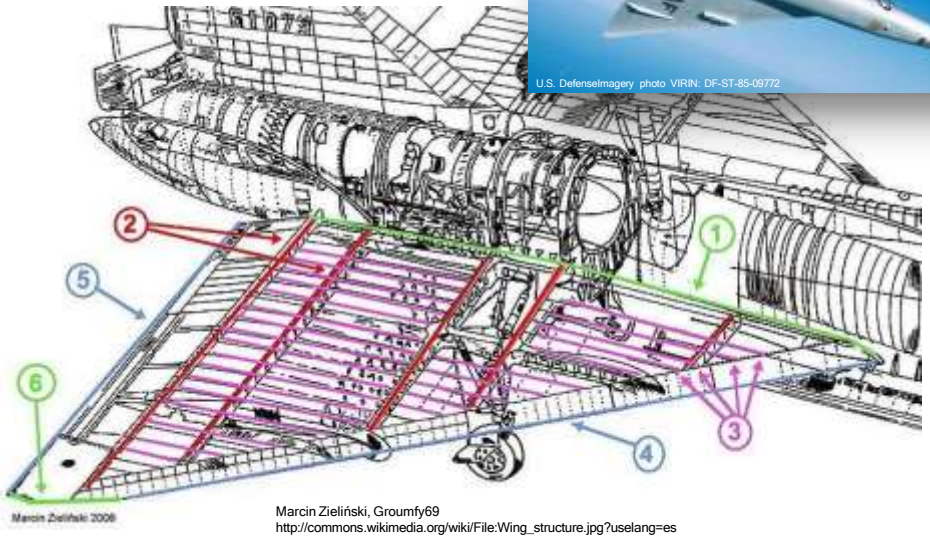
- **Spars**

Convair F-106A Delta Dart

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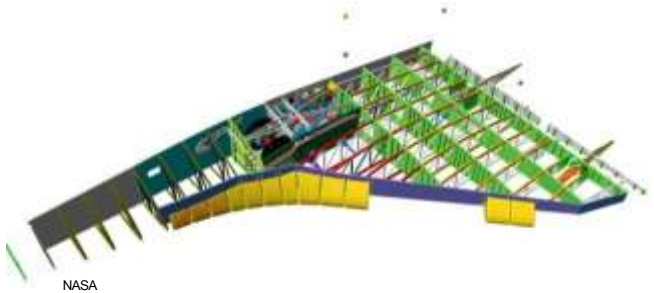


U.S. Defenselmagery photo VIRIN: DF-ST-85-09772




Marcin Zieliński, Groumfy69
http://commons.wikimedia.org/wiki/File:Wing_structure.jpg?uselang=es

**Torsion box structure: multispar**
**Parallel spars**



NASA
http://commons.wikimedia.org/wiki/File:Shuttle_Left_Wing_Cutaway_Diagram.jpg?uselang=es
http://www.nasa.gov/columbia/home/COL_wing_diagrams.html

(1) Wing root
(2) Spars
(3) Ribs
(4) Leading edge
(5) Trailing edge
(6) Wing tip



**NASA Photo ID:** S81-30746
http://commons.wikimedia.org/wiki/File:Columbia_landing_on_Rogers_dry_lake.triddle.jpg

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Wing structure

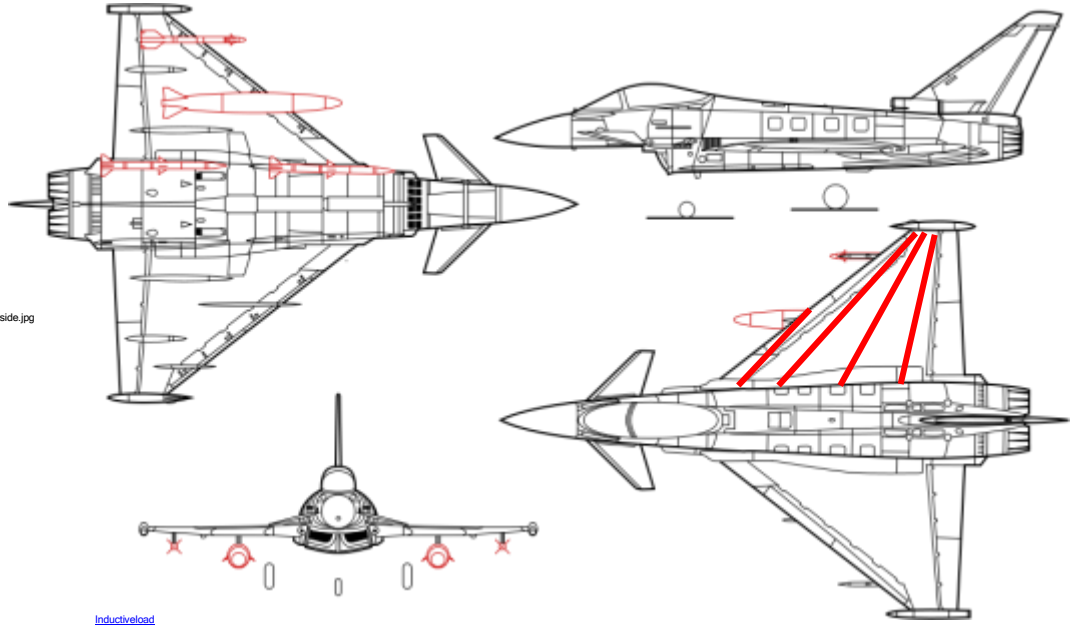
- **Spars**



tMH
ttp://commons.wikimedia.org/wiki/Category:Eurofighter_Typhoon?uselang=es#mediaviewer/File:Eurofighter_Typhoon,_Underside.jpg

Eurofigther

**Torsion box structure: multispar**
**Converging spars**



Inductiveload

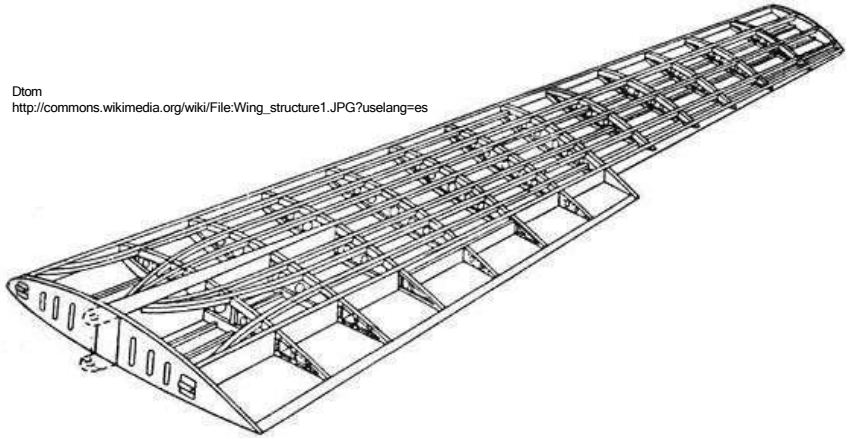
http://commons.wikimedia.org/wiki/File:Eurofighter_Typhoon_line_drawing.svg?uselang=es

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# Wing structure

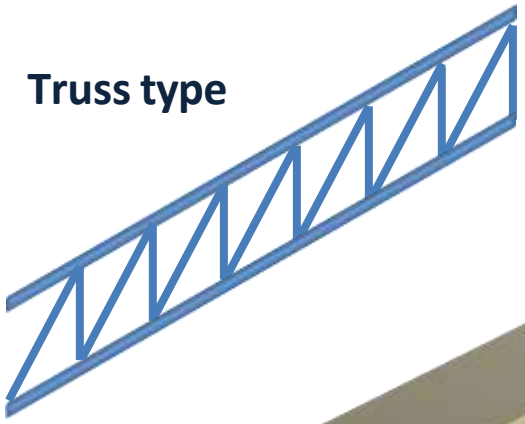
- **Spars**

## Types of spar construction

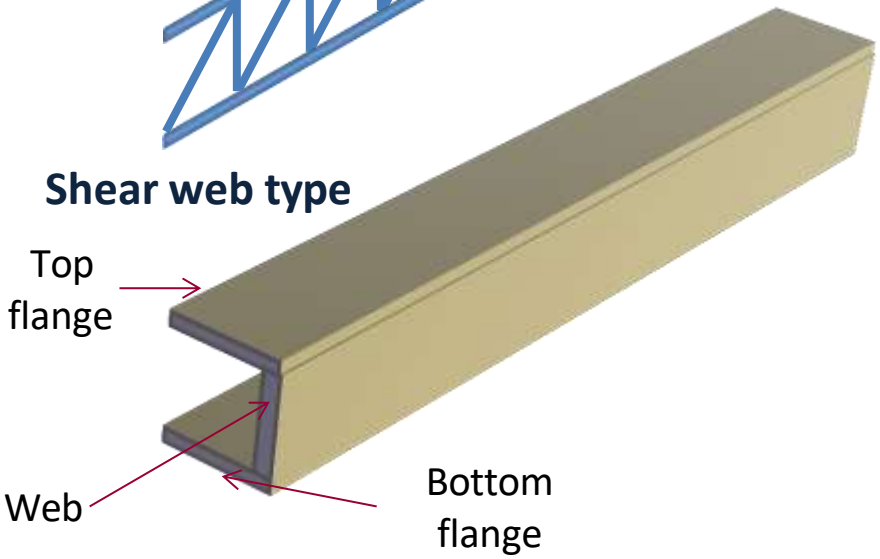


Dtom
http://commons.wikimedia.org/wiki/File:Wing_structure1.JPG?uselang=es

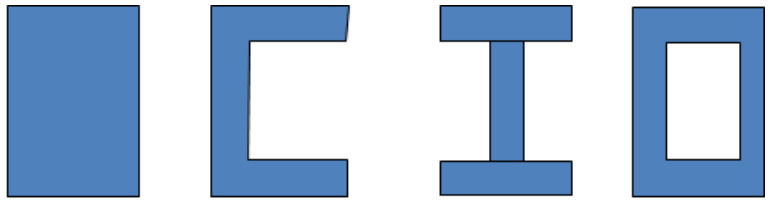
### Truss type



### Shear web type



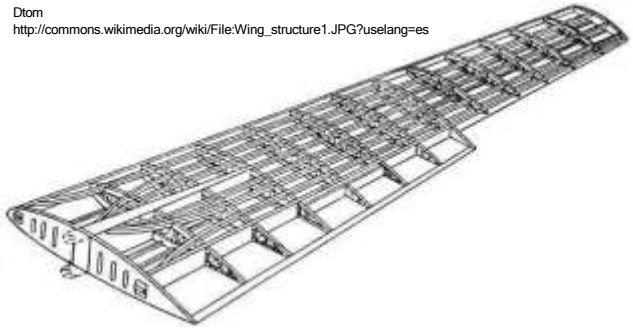

## Cross-section geometries



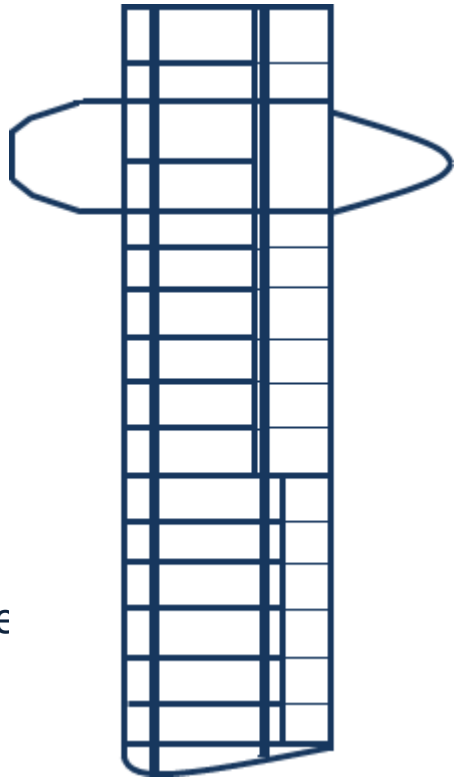

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Wing structure

- **Ribs**



Dtom
http://commons.wikimedia.org/wiki/File:Wing_structure1.JPG?uselang=es






http://commons.wikimedia.org/wiki/File:Canadair_CL-215-1A10_CL-215-I_AN0786306.jpg?uselang=es
Photo: Pedro Aragão

The wing ribs determine the shape and thickness of the wing

Ribs types

-Shear web ribs
- Truss ribs

The rib spacing is selected after a optimization process (skin+stringer+ribs)

Rib are to be located at each aileron and flap hinge.

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Wing structure

- **Ribs arrangement**



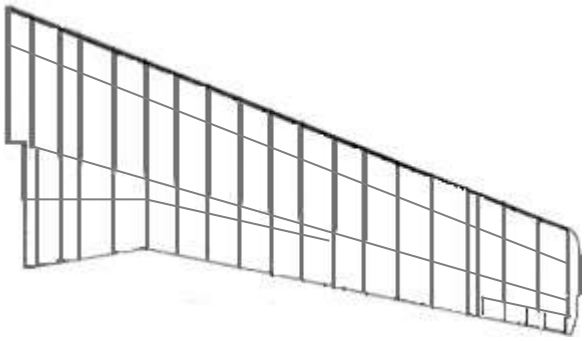

http://commons.wikimedia.org/wiki/File:Aerospatiale_SE-210_Caravelle_10B3_Super_B,_Sterling_Airways_AN0018394.jpg?uselang=es

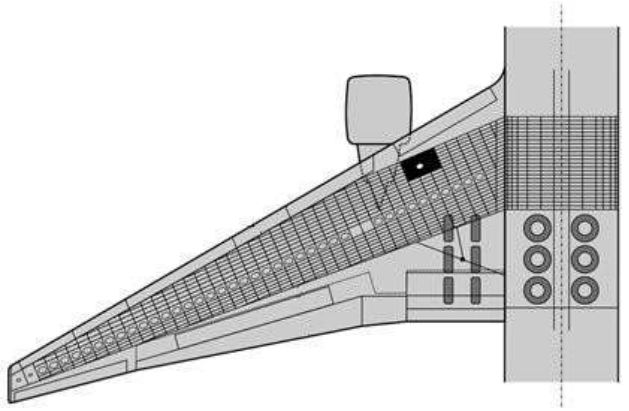
**Aerospatiale Caravelle**




http://commons.wikimedia.org/wiki/Boeing_767#mediaviewer/File:LOT_Polish_Airlines,_Boeing_767-25D_ER,_Mississauga_(295649204).jpg

**Boeing 767**





Tosaka
http://commons.wikimedia.org/wiki/File:Jetliner%27s_wing_structure_(B-777).PNG

Wing ribs parallel to the flight path

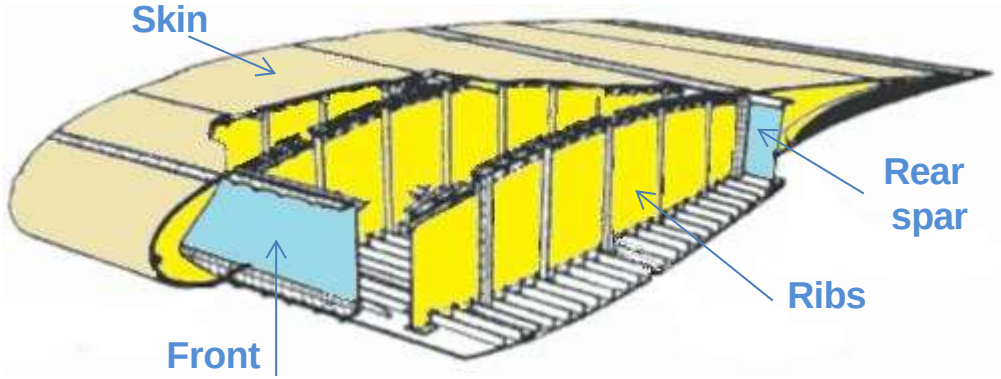
Rib perpendicular to the spars

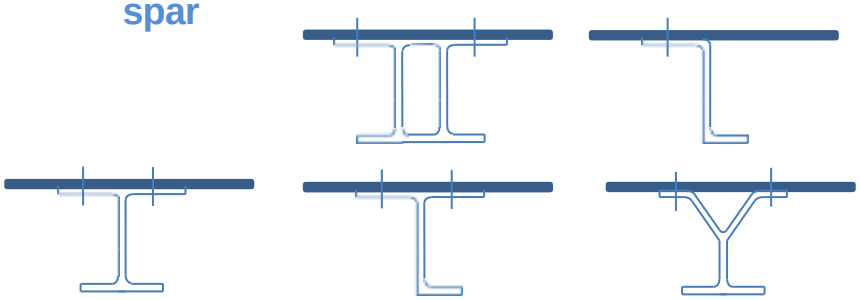
**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Wing structure

## Skin

50-70% of the structural weight of the wing



**Typical wing skin-stringer panels**

It transmits aerodynamic forces to the longitudinal and transverse members

In some designs the spar caps resist the bending stresses, in other skin is the bending-load resistant element

**Integrally stiffened panels**

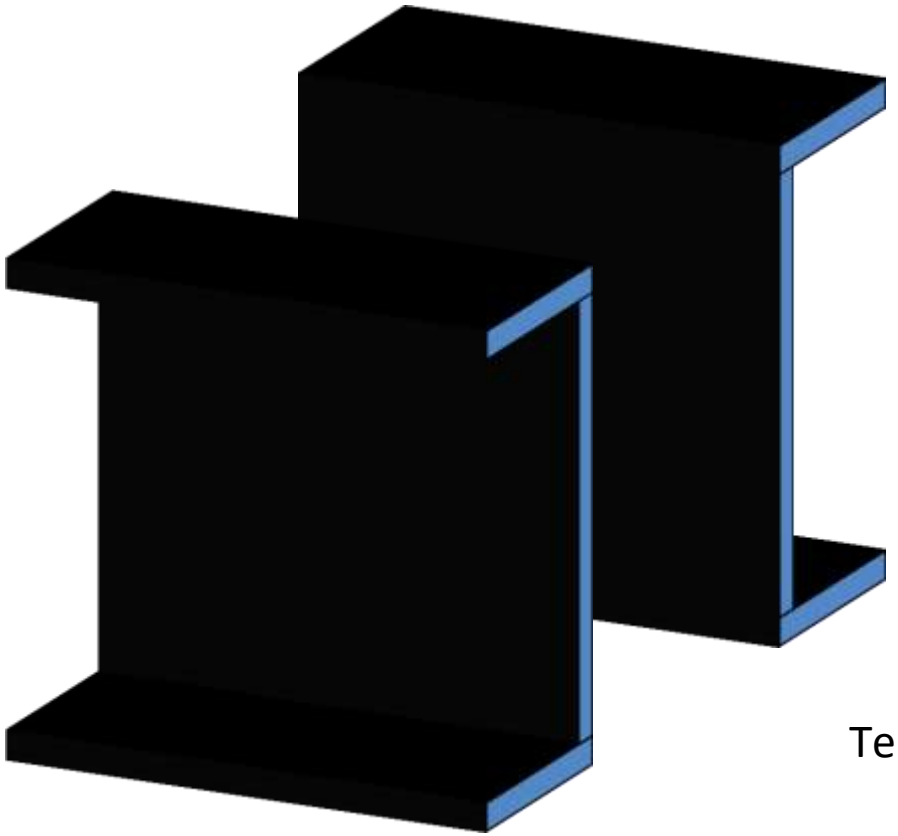


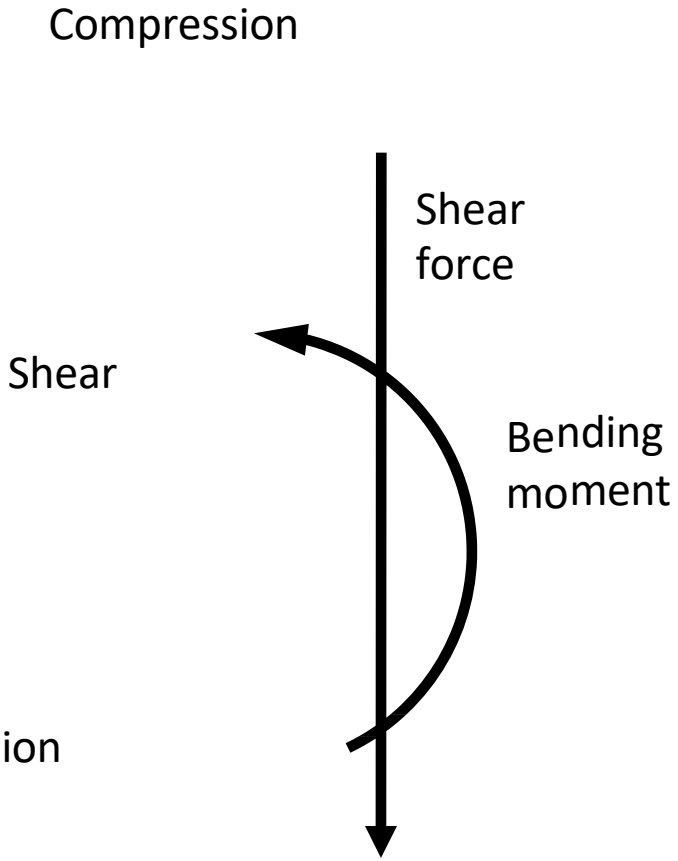
Wing skins are machined from a thick plate to obtain the skin-stringer geometry

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Wing structure

- **Wing box structure**

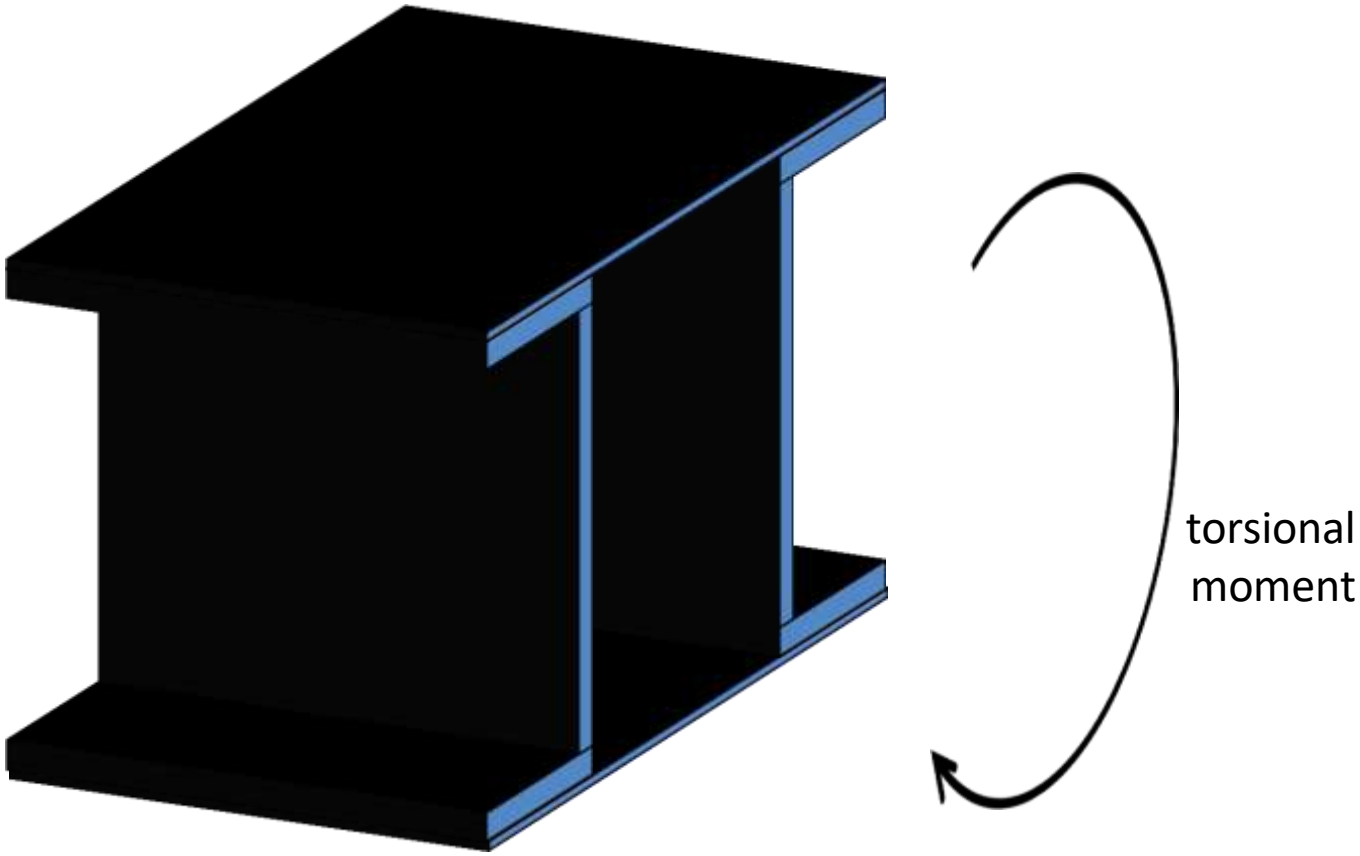





Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Wing structure

- **Wing box structure**




Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

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# Fuselage structure

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Fuselage structure

## Fuselage functions







The fuselage function is to carry the payload and support many the aircraft system. Thy connect the other structural element (wing and empennage)

Loads on the fuselage:

- Wing reactions
- Landing gear reactions
- Internal pressure

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Fuselage structure

## Fuselage types

### Non stressed-skin structures

- Truss fuselage (1)
- Geodetic fuselage (2)

### Stressed-skin Structures

- Corrugated (3)
- Monocoque fuselage (4)
- Semi-Monocoque fuselage (5)

**Blériot XI**




http://commons.wikimedia.org/wiki/Bl%C3%A9riot_XI?uselang=es#mediaviewer/File:Bleriot_XI_Thulin_2.jpg

**Vickers Wellington**




http://commons.wikimedia.org/wiki/File:Vickers_Wellington_Mk2.jpg

**Ford Trimotor**

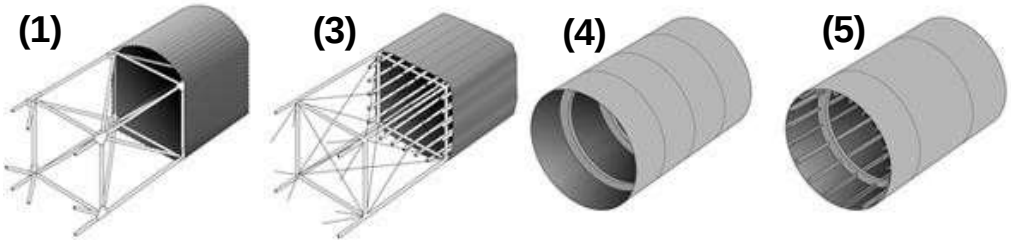



http://commons.wikimedia.org/wiki/File:Ford_Trimotor_EAA.jpg?uselang=es

**Rutan VariEze**




http://commons.wikimedia.org/wiki/Category:Rutan_VariEze#mediaviewer/File:VariEze_in_flight.jpg




Author: Tosaka
http://commons.wikimedia.org/wiki/File:Airframe_(4_types).PNG?uselang=es

**Airbus A350**




http://commons.wikimedia.org/wiki/Airbus_A350#mediaviewer/File:A350_First_Flight_-_Low_pass_03.jpg

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Fuselage structure

- **Fuselage types**

## Truss fuselage




http://commons.wikimedia.org/wiki/File:Sopwith_Camel_taking_off,_Masterton,_New_Zealand,_April_2009.jpg?usela

Sopwith Camel, 1917

Most of the early aircraft used this kind of construction




The diagonal elements can be bars or wires

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Fuselage structure

## ❑ Fuselage types

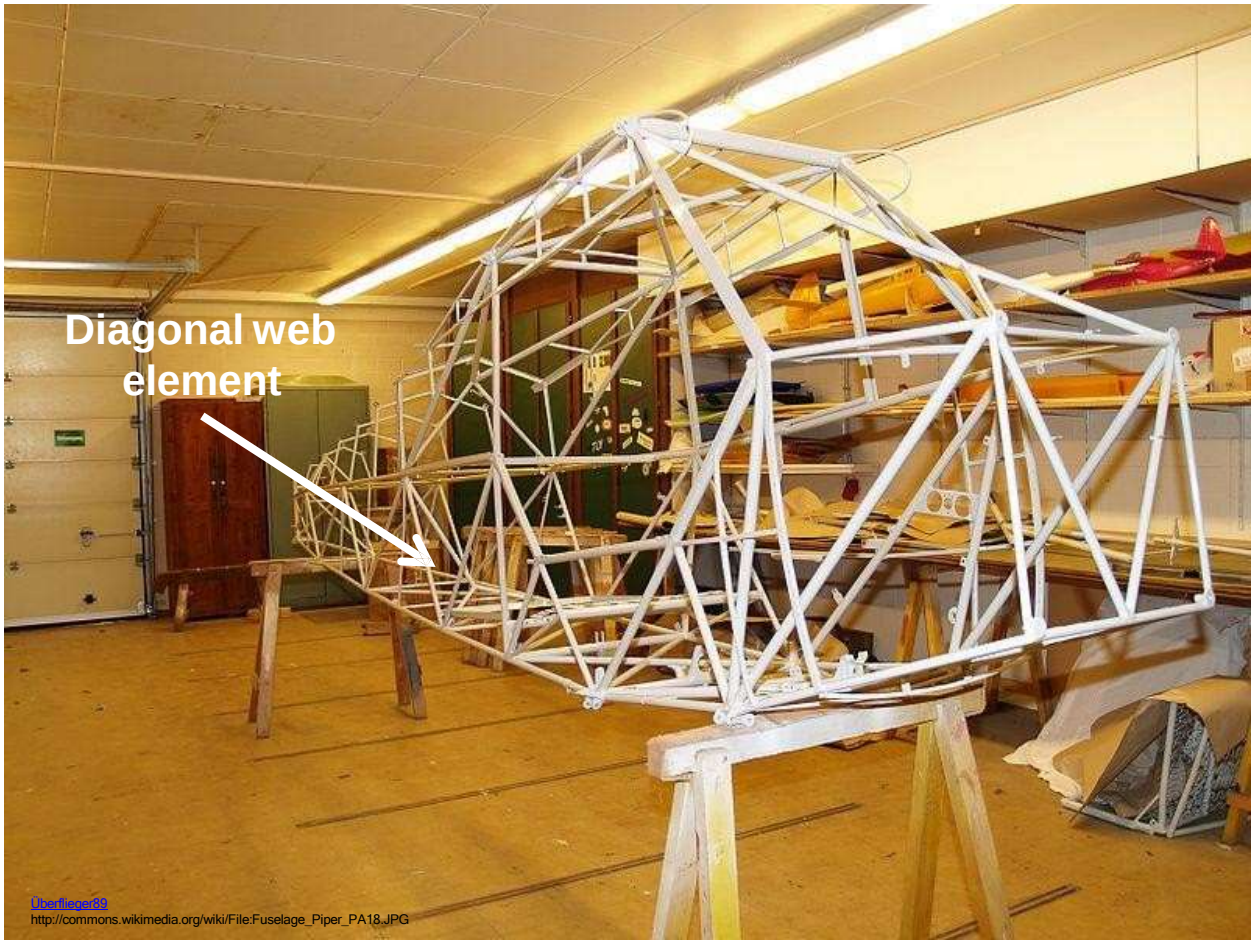
### Truss fuselage




http://commons.wikimedia.org/wiki/Category:Super_18_Model_S18#mediaviewer/File:Super18-Super18-180.jpg
Photo: FlugKerl2

Piper PA-18

This type of structures are still used in small aircraft




Überflieger89
http://commons.wikimedia.org/wiki/File:Fuselage_Piper_PA18.JPG

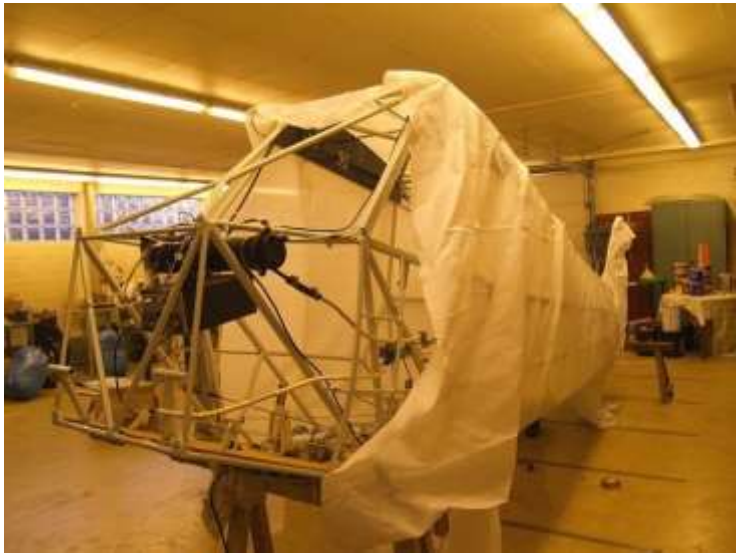
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# Fuselage structure

## ❑ Fuselage types

### Truss fuselage

Non structural skin



http://commons.wikimedia.org/wiki/Piper_PA-18#mediaviewer/File:L18FuselageCUT.jpg

Überflieger89



http://commons.wikimedia.org/wiki/Piper_PA-18#mediaviewer/File:L18FuselageCT.jpg

Überflieger89

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# Fuselage structure

## ❑ Fuselage types

### Geodetic fuselage




Vickers Wellington
1936




**Advantages:**
Lightweight structure
High strength
Damage tolerance

**Disadvantages:**
Complexity



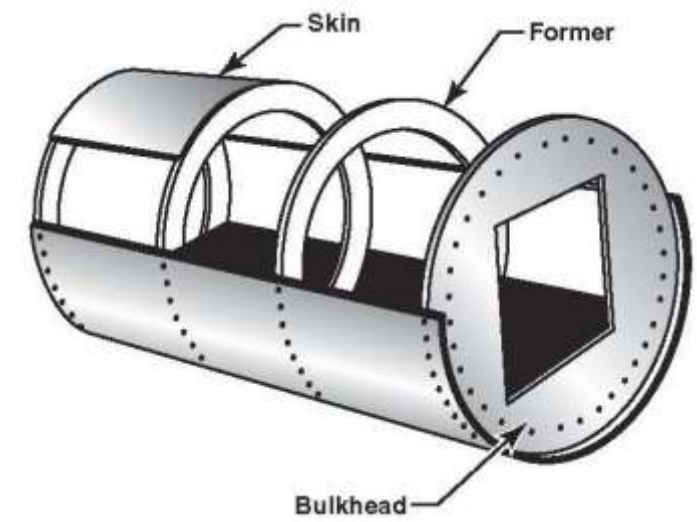

In this type of construction multiple flat strip stringers are wound about the formers in opposite spiral directions

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# Fuselage structure

## ❑ Fuselage types

Eas4200c.f08.nine.o
http://commons.wikimedia.org/wiki/File:Chapter_1_img_26.jpg?uselang=es




The skin supports the primary stresses



Deperdussin monoplane racer 1912

First monocoque design

Designed by Louis B`echereau

### Monocoque fuselage

Virtually no internal framework

Only transversal elements (former) + skin

Stoddard-Hamilton Glasair III




http://commons.wikimedia.org/wiki/Category:Stoddard-Hamilton_Glasair_III#mediaviewer/File:GlasairIII-Landing.jpg

Used today in small aircrafts with complete composite structure

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Fuselage structure

## ❑ Fuselage types




http://commons.wikimedia.org/wiki/File:Ju-Air_Junkers_Ju-52-3m_HB-HOS_Hahnweide_2011_03.jpg?uselang=es

Junkers Ju-52/3m
1932



Ford Tri-Motor
1926




http://commons.wikimedia.org/wiki/File:Ford_Tri-Motor_(EAA)_(340406538).jpg?uselang=es




http://commons.wikimedia.org/wiki/File:Ju52_civil.jpg?uselang=es

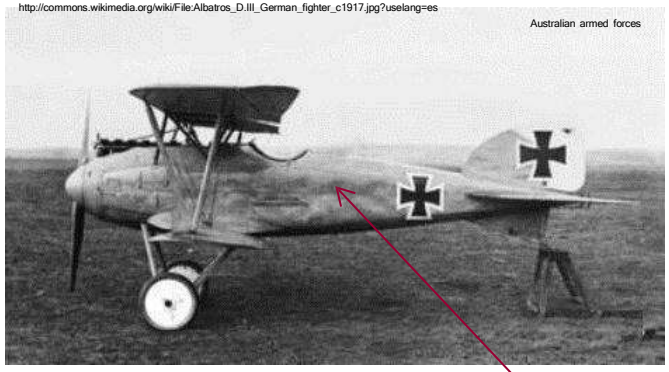
It had a structure with metallic corrugated skin

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# Fuselage structure

## ❑ Fuselage types




Albatros D.III
1916

**Streamlined fuselage**

### Semi-Monocoque fuselage

**Longitudinal elements**

- Longerons
- Stringers

**Transverse element**

- Frames
- Bulkheads




**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Fuselage structure

## ❑ Skin and stringer

The skin carries the shear stresses produce by the internal forces (torque and shear force)




**Typical fuselage skin-stringer panels**



B-247



C-141
C-5



B-707
B-720 DC-10
B-727 Trident
B-737
B-747

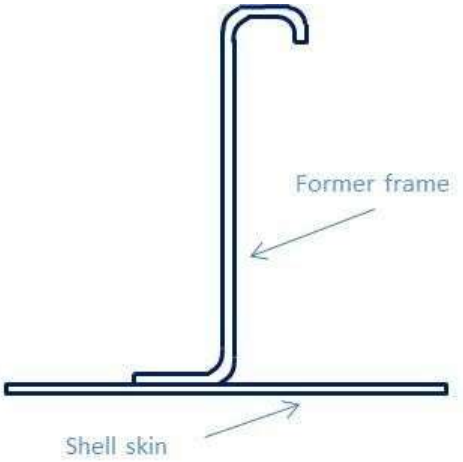


L-188 L-1011
CV-880 A300
F-28

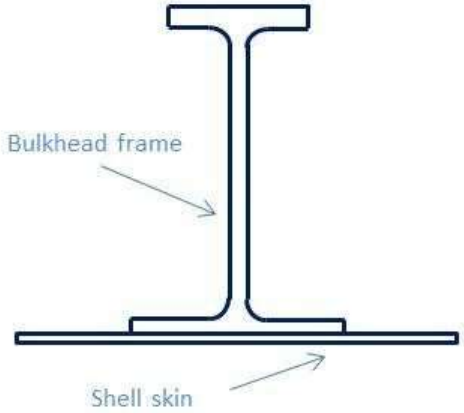
**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Fuselage structure

## ❑ Frames




Former frame




Bulkhead

Author: Kolossos
http://commons.wikimedia.org/wiki/File:Fuselage-747.jpg?uselang=es



Distance between frames are around 500 mm

Frame thickness are 75-150 mm

Frames function

- ❑ Maintain the shape of the fuselage
- ❑ Increase the buckling strength of the stringer



Bibi95
http://commons.wikimedia.org/wiki/File:Exposition_-_Les_100_ans_de_l%27a%C3%A9rospatiale_-_Paris_-_4_Octobre_2008_(2913799759).jpg

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Fuselage structure

## ❑ Frames

Bibi95
http://upload.wikimedia.org/wikipedia/commons/1/19/Exposition__Les_100_ans_de_l%27a%C3%A9rospatiale_-_Paris_-_4_Octobre_2008_%282913796295%29.jpg



Front dome

Rear dome

Frames



NASA
http://commons.wikimedia.org/wiki/File:Shuttle_Carrier_Aircraft_interior_bulkhead.jpg

Types of frames

- Former frame
- Bulkhead
- domes

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# Fuselage structure

## ❑ Cutout

The fuselage has requirements for opening cut-out such windows, doors, service panels, hatches, bomb bays...




The direct load paths are interrupted and as a result the structure around the cut-out must be reinforced to maintain the required strength




**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Stabilizers structure

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# Stabilizers structure

## ❑ Stabilizers

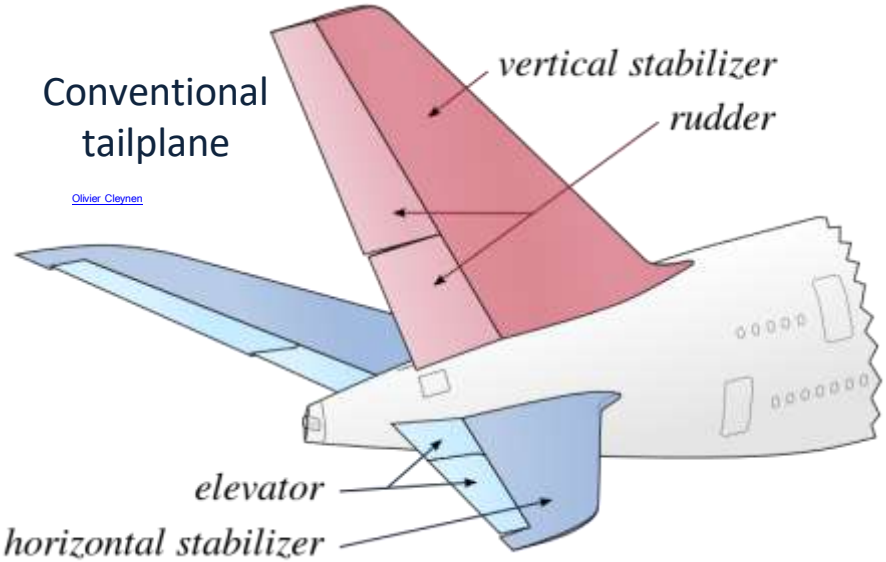



Alternate tailplane: V-tail




Horizontal stabilizer of an Airbus A320

Conventional tailplane




http://commons.wikimedia.org/wiki/File:Tail_of_a_conventional_aircraft.svg?uselang=es

Stabilizer construction is similar to wing construction.

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Stabilizers structure

## ❑ Horizontal Stabilizers




http://commons.wikimedia.org/wiki/File:Boeing_707-131B,_Trans_World_Airlines_-_TWA_AN2228670.jpg?uselang=es

Box construction with spars, closer light rib spacing (less than 250 mm) and skin without stringer

Low cost and high torsional stiffness

**Boeing B07**

Box construction with spar, stronger ribs and skin with stringer

Low weight




http://commons.wikimedia.org/wiki/Lockheed_L-1011#mediaviewer/File:Lockheed.tristar.flying.arp.jpg

**Lockheed L-1011**

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Stabilizers structure

## ❑ Horizontal Stabilizers




http://commons.wikimedia.org/wiki/File:F-16_Falcon_Fighting_-_091119-F-7323C-292.jpg?uselang=es

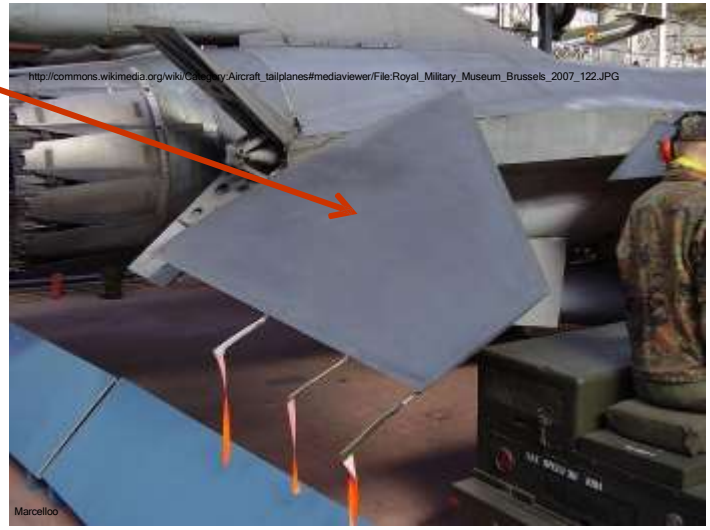
**Lockheed Martin F-16**

In fighter design, usually no spar + ribs construction is used

Two designs

a) Honeycomb + one spar

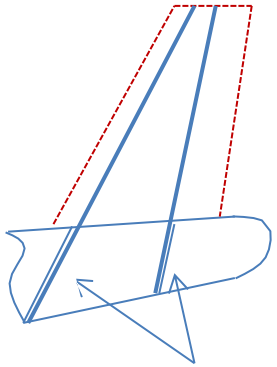
b) Corrugation multi-spar



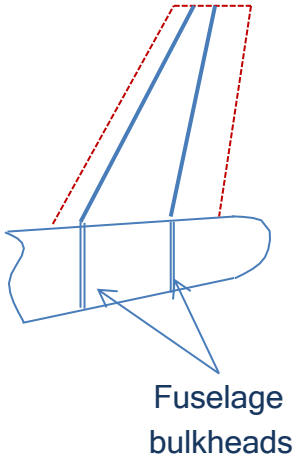

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Stabilizers structure

## ❑ Vertical Stabilizers




Tail box front and rear spar terminated at aft fuselage bulkhead




Tail box front and rear spar terminated outside of fuselage




http://commons.wikimedia.org/wiki/Category:Aircraft_tails#mediaviewer/File:Airforce_Museum_Berlin-Gatow_322.JPG



Russavia
http://commons.wikimedia.org/wiki/File:N551VL_Gulfstream_G550_G-V-SP_(9489975740).jpg

The span of the t-tail fin is aprox. 1/3 than conventional tail

The vertical stabilizer must be made considerably stronger and stiffer to support the forces generated by the tailplane

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# References


- **Aircraft Structures for engineering students**
  H.G. Megson
  Elsevier
  Section B1 Principles of stressed skin constuction
  Chapter 12 Structural components of aircraft
  Section 12 B2 Airworthisess and airframe loads
  Chapter 13 Airworthiness
  Chapter 14 Airframe loads

- **Airframe Structural Design**
  M.C.Y. Niu
  Hong Kong Conmilit Press LTD
  Section 3. Aircraft loads
  Chapter 8 Wing Box Structure
  Chapter 11Fuselage



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# Aerospace Structures

## Chapter 1.
## Structural description of the aircraft

### Structures in the aeronautical sector

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

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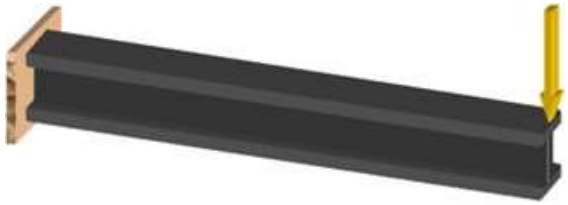
# CHAPTER 1. Structural description of the aircraft

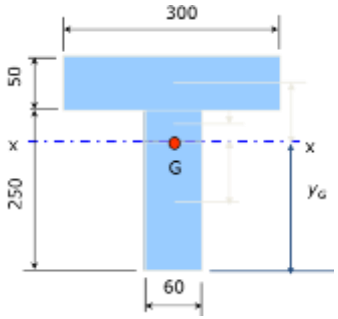
## Structures in the aeronautical sector

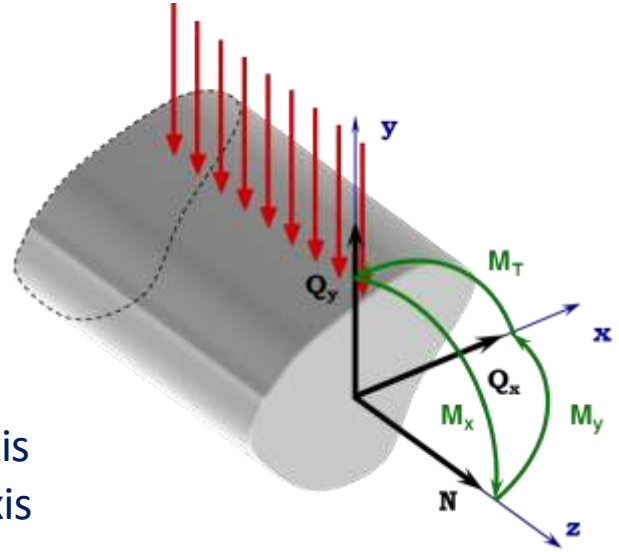

- Introduction
- Truss structures
- Frame structures
- Lattice structures
- References


Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

## ❑ Strength of Materials



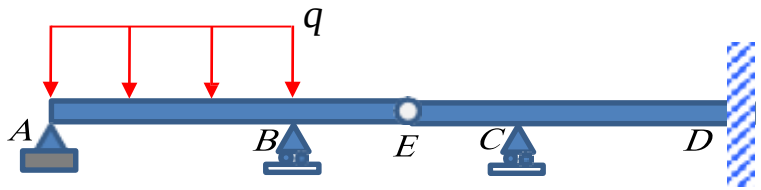



$$\frac{du_z^o(z)}{dz} = \frac{N}{EA}$$
$$\frac{d\theta_x(z)}{dz} = \frac{M_x}{EI_x}$$
$$\frac{d\theta_y(z)}{dz} = \frac{M_x}{EI_y}$$

EA: Axial stiffness
EIx: Bending stiffness around x axis
EIy: Bending stiffness around y axis

$$\sigma_{zz}(x, y, z) = \frac{N}{A} + \frac{y \cdot M_x}{I_x} - \frac{x \cdot M_y}{I_y}$$




Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

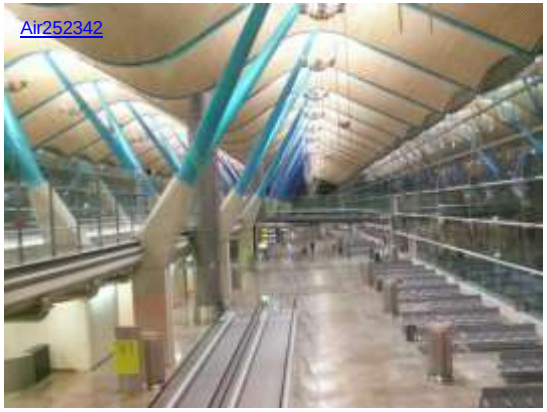
# ❑ Airport and hangar structures

Examples of truss/beam structures




https://commons.wikimedia.org/wiki/File:Shenzhen-airport-terminal.JPG?uselang=es

Shenzhen-airport-terminal




https://commons.wikimedia.org/wiki/File:Barajas_interior2.jpg?uselang=es

Terminal 4 Madrid Airport




https://upload.wikimedia.org/wikipedia/commons/2/21/US_Navy_030302-A-8380W-003_Seabees_build_hangar.jpg




https://commons.wikimedia.org/wiki/Category:Hangars#/media/File:Hangaretes_do_Projeto_32_-_Foto_Set_2014.jpg

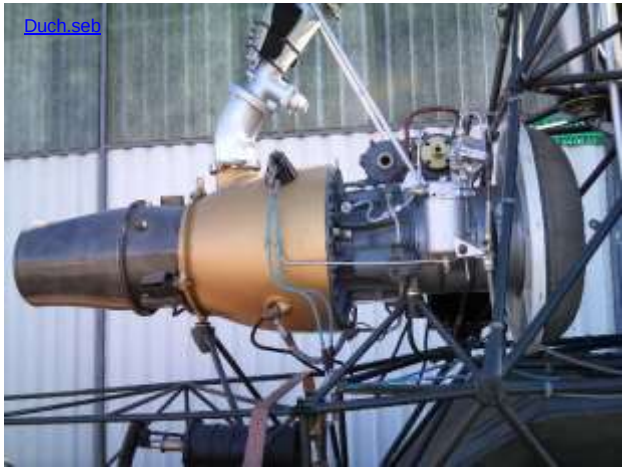
Hangar structures

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Introduction

## ❑ Beam structures in aircrafts

Examples of truss/beam structures




https://upload.wikimedia.org/wikipedia/commons/c/c0/Turbomeca_Palouste_%28MAA%29.JPG?uselang=es

Engine mount structure




http://commons.wikimedia.org/wiki/File:Fuselage_Piper_PA18.JPG

Truss structure fuselage




https://upload.wikimedia.org/wikipedia/commons/3/3d/A320-main-landing-gear.jpg?uselang=es

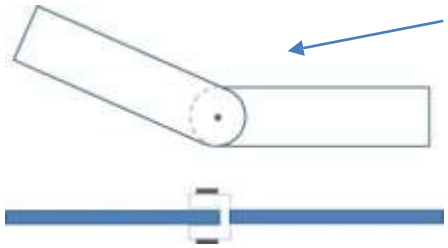
Landing gear structure

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Truss structures

## ❑ Definition: Hinged structures

The joints between bars allow relative rotation about an axis perpendicular to the plane of the structure.







https://upload.wikimedia.org/wikipedia/commons/7/7f/Terminal_%285791945861%29.jpg

Fukuoka Airport - International Terminal




https://upload.wikimedia.org/wikipedia/commons/7/74/T5_Trusses_4.jpg

https://upload.wikimedia.org/wikipedia/commons/7/75/London_Heathrow_T5_AB2.JPG

London Heathrow Terminal 5

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Truss structures

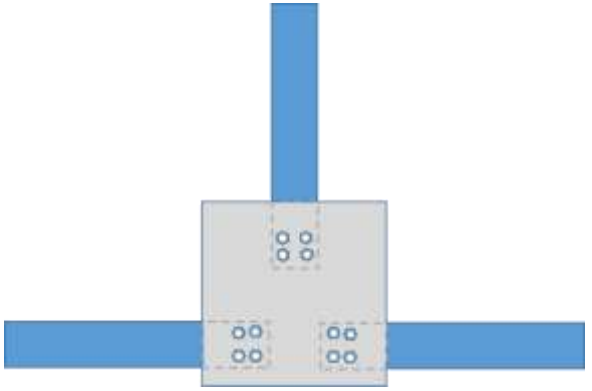
## ❑ Definition



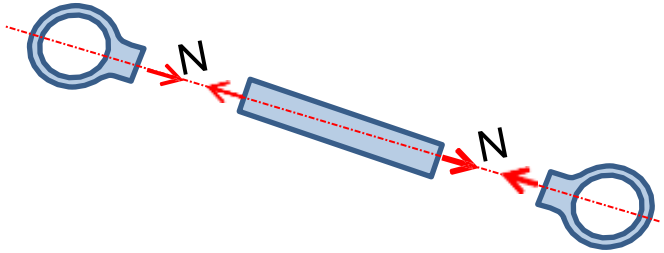

https://commons.wikimedia.org/wiki/File:THROUGH_TRUSS_VERTICAL-DIAGONAL_INTERSECTION_DETAIL_-_Richmond_Bridge,_US_90A_at_Brazos_River,_Richmond,_Fort_Bend_County,_TX_HAER_TEX,79-RICH,2-25.tif?uselang=es

Richmond Bridge

Thin plate joint



The joint can be done by screws, rivets or welding.



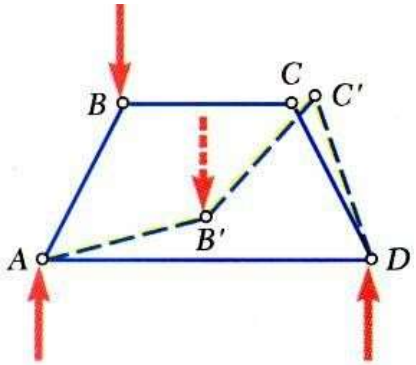

Only axial forces appear in the bars

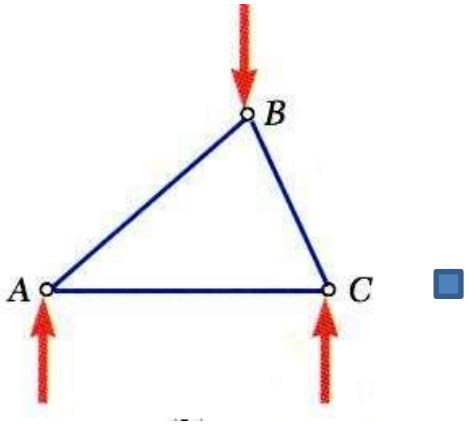
**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Truss structures

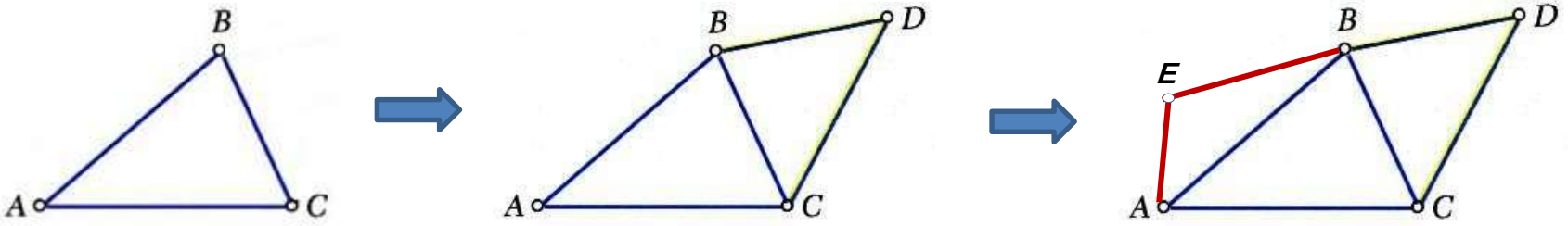
## ❑ Definition

- The structure must be stiff and avoid the collapse under the application of loads.




- The simplest stable structure is the triangle.
- Stable structures can be design adding two bars and one node.

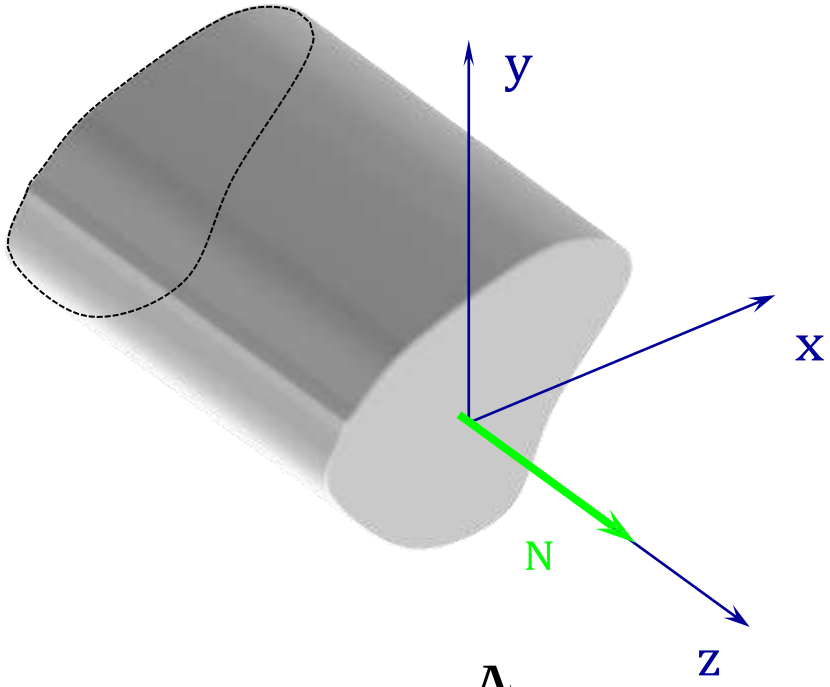



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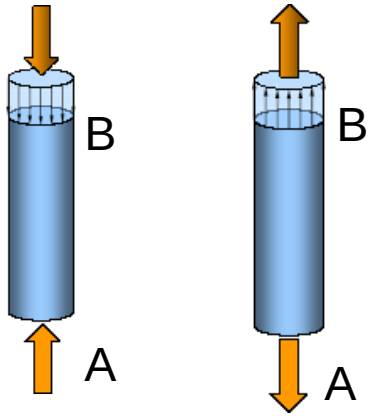
# Truss structures

## ❑ Internal forces

**Axil**




$$\sigma_z = \frac{N}{A}$$

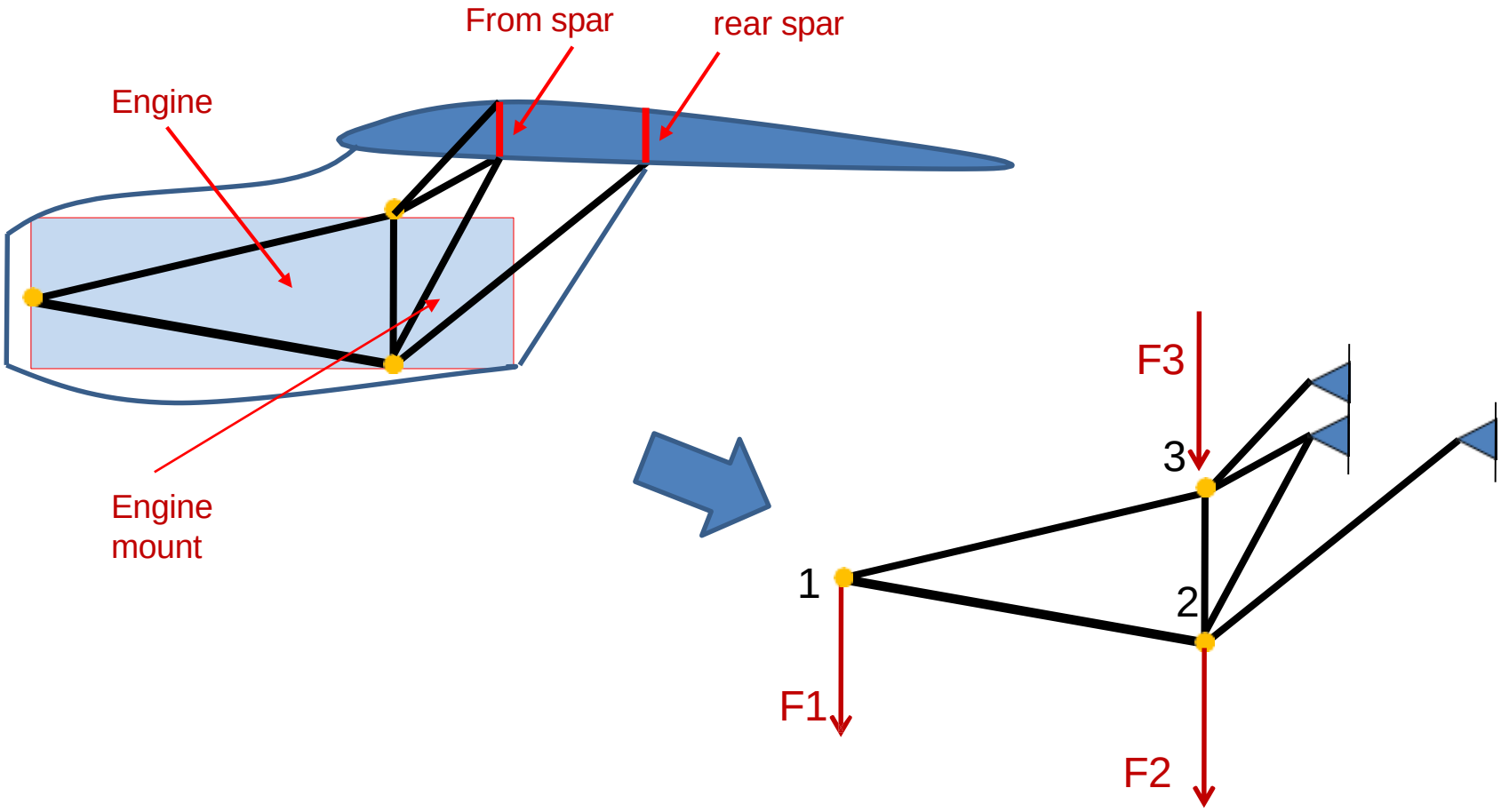



$$w^B - w^A = \int_A^B \frac{N}{EA}\, dz$$

Axial members are used to carry extensional or compressive loads applied in the direction of the axial direction of the member

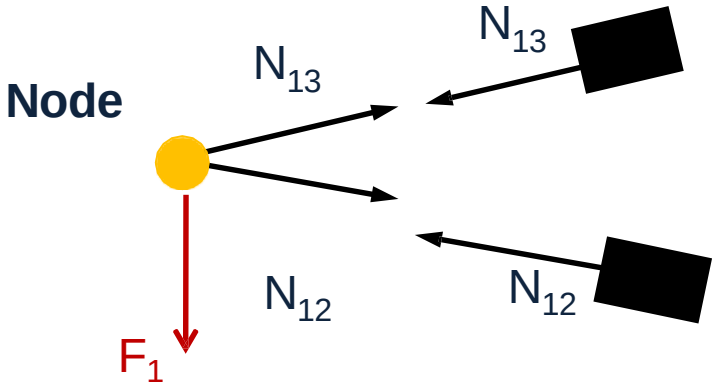
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

## ❑ Analysis of truss structures: Example

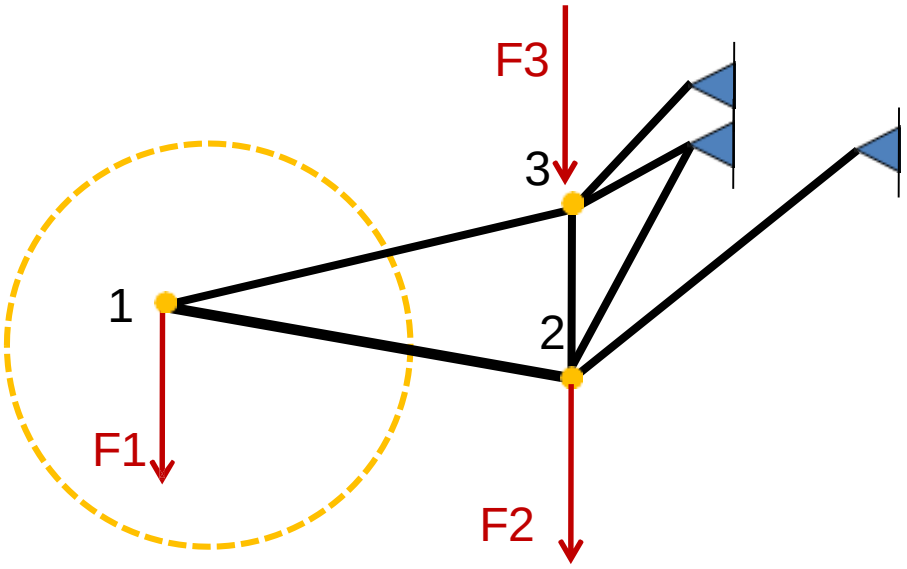



Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

## ❑ Analysis of truss structures: Example




The axial forces in the bars can be calculated by equilibrium forces in each of the nodes of the structure and supports (in statically determinate structures)




$$\sum F_H = 0$$

$$\sum F_V = 0$$

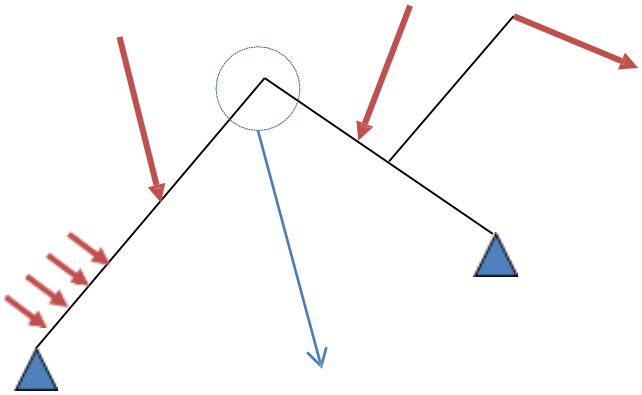
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Frame structures

## ❑ Definition

A frame is a structure having the bars joined via fixed joints

Each bar is subjected to internal axial forces, internal shear forces and bending moments







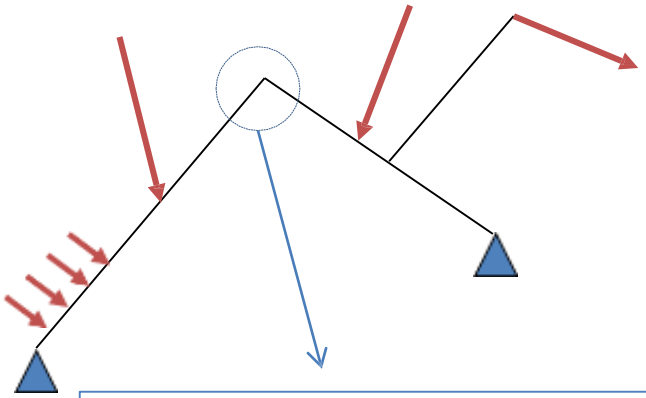
https://www.flickr.com/photos/elycefeliz/6489247171/in/photolist-aTr66V-9o8VGk-9NEwpU-dFgHej-rp8qXm-9NGKrb-8f4Htd-nKV2Yb-nKXpv6-5oCDR6-opWwJy-ebwmMy-oKSEUg-oEpw9U-8T9tUk-bzbKEW-3pFFhR-jEfb7X-9UmsB9-7rwjBL-e6BEva-jEhbsL-9NDZCz-e79FUE-6det9Z-p9zZLq-oxm5b2-cBUevf-kaZNV8-pJrjtK-9NH11D-e4wfhJ-oPP4Zy-oqAqRA-fzbsUG-s9moow-e6UeVm-opWKGB-fzbqZ5-23wkna-fyWbf6-hvwFV4-699PPa-42GH8K-6NyHgr-9NyoX4-dcPcKf-9NDtMv-a3W2c4-oXi8v1

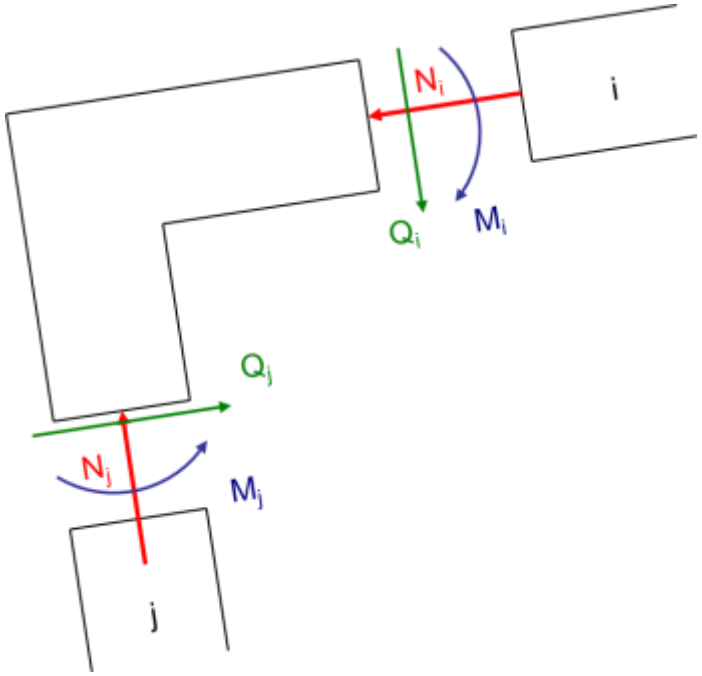
**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

## ❑ Definition

A frame is a structure having the bars joined via fixed joints

Each bar will be subjected to internal axial forces, internal shear forces and bending moments

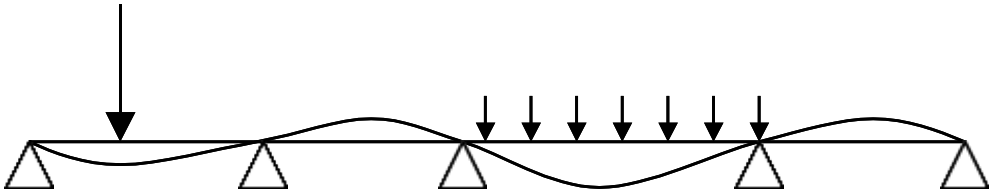




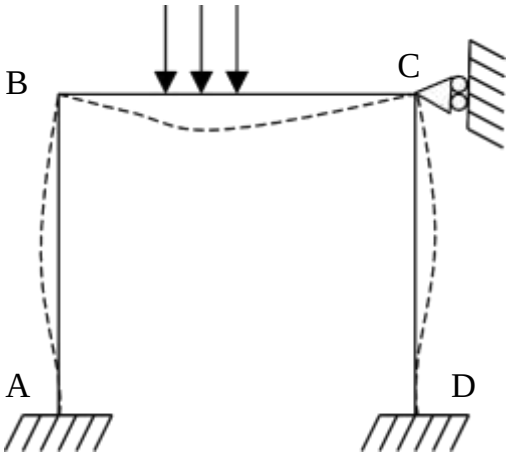

The nodes will have equilibrium under the influence of internal axial forces, internal shear forces, bending moments and eventual external loads which may act on the structure

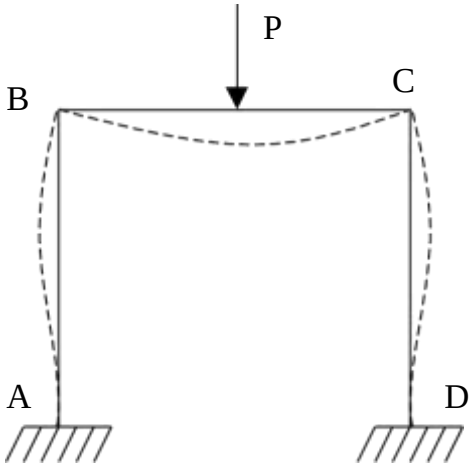
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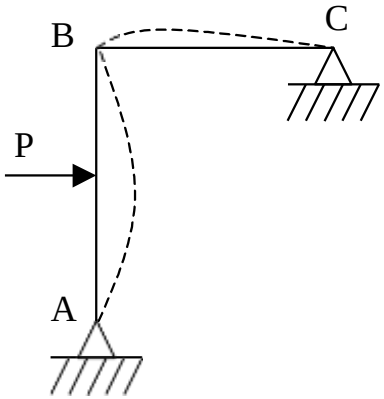
## ❑ Non-traslational frames



The nodes of a non-translational structure does not move but the sections of the bars can turn



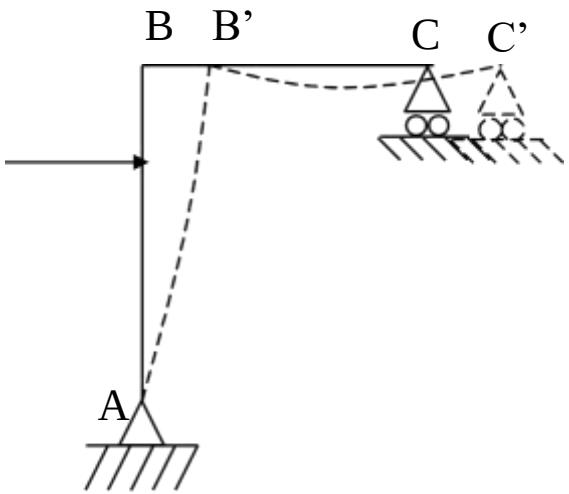



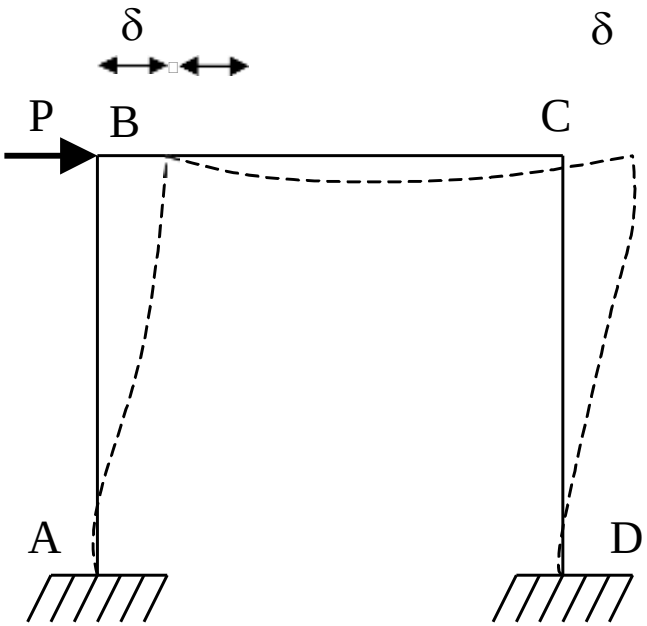

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## ❑ Non-traslational frames

WHEN A STRUCTURE IS TRASLATIONAL?

The nodes of translational structures the nodes can move, this is considered the unknown of the problem

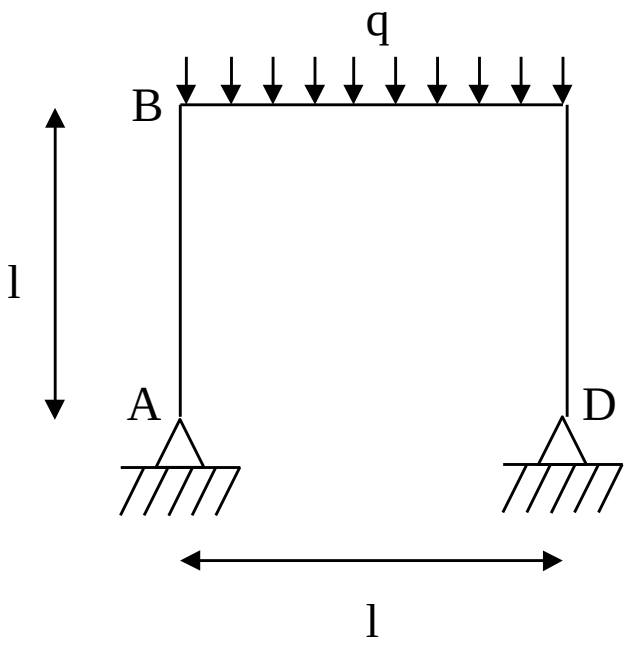


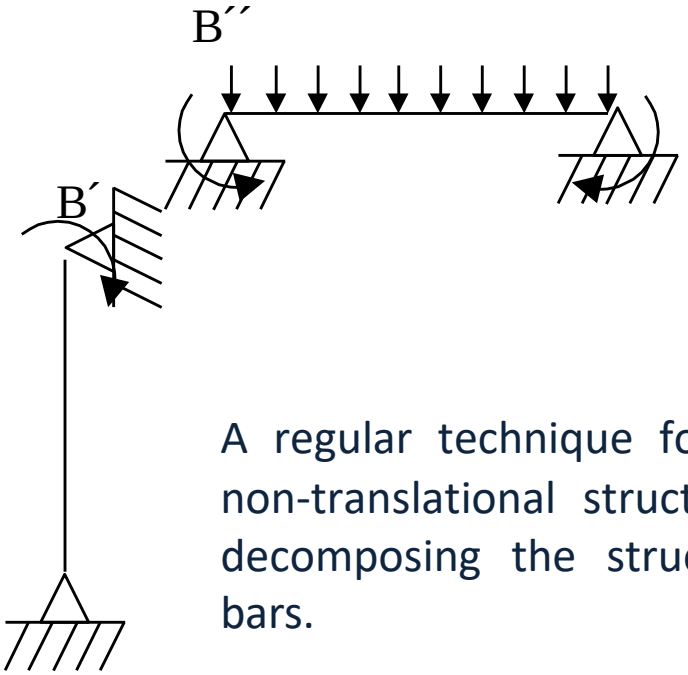




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# Frame structures

## ❑ Non-traslational frames



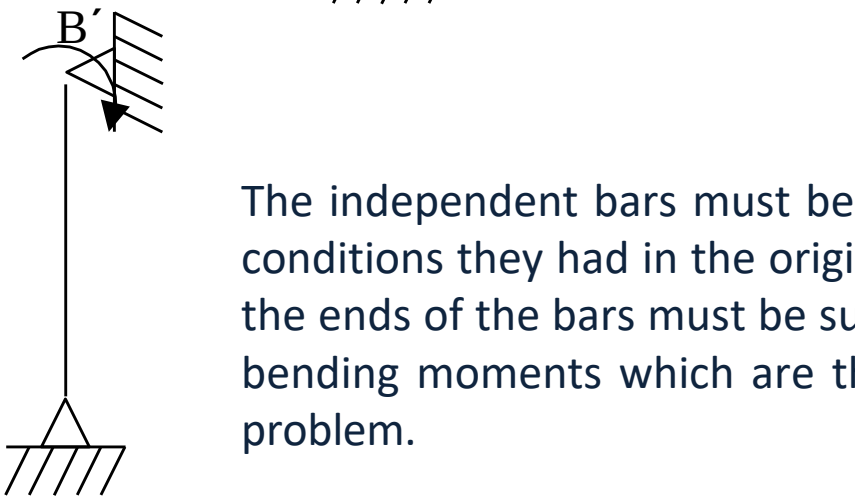




A regular technique for calculation of non-translational structures consist on decomposing the structure in several bars.

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

## ❑ Non-traslational frames

The bars, after the decomposition of the structure, must be constrained at their ends via supports.



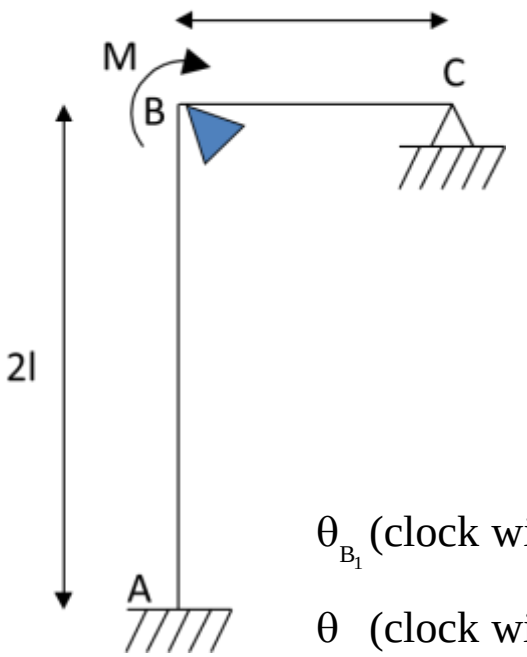

The independent bars must be subjected to same conditions they had in the original structure. Thus, the ends of the bars must be subjected to external bending moments which are the unknown of the problem.

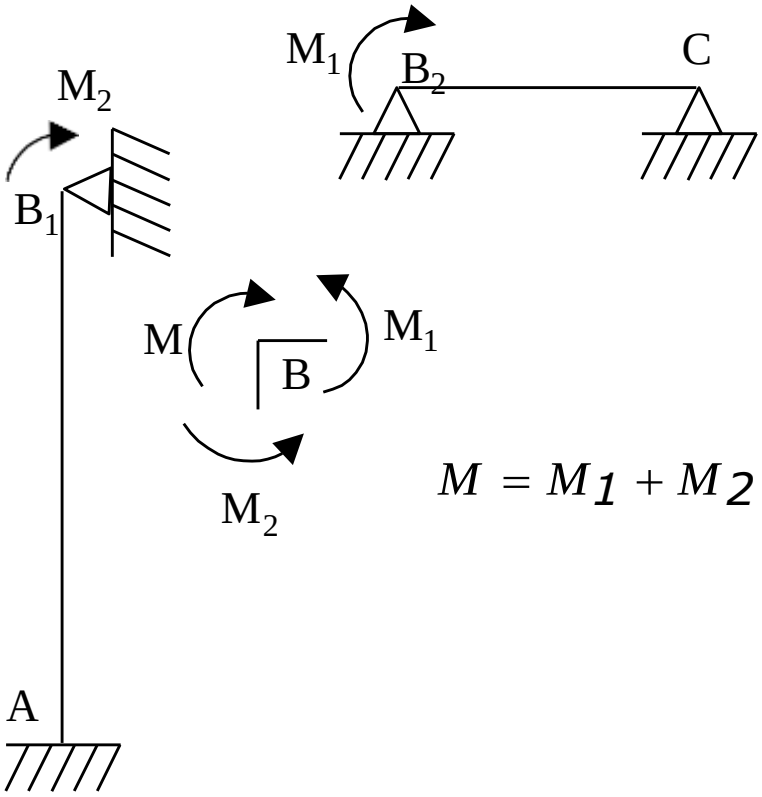
The compatibility condition is “rotations of the end sections of bars that converge at a node must be identical»

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

## ❑ Non-traslational frames

Example




$$M = M_1 + M_2$$

$$\theta_{B_1}(\text{clock wise}) = \theta_{B_2}(\text{clock wise})$$

$$\theta_{B_1}(\text{clock wise}) = \frac{M_2 (2l)^2}{4EI(2l)} = \frac{M_2 l}{2EI}$$

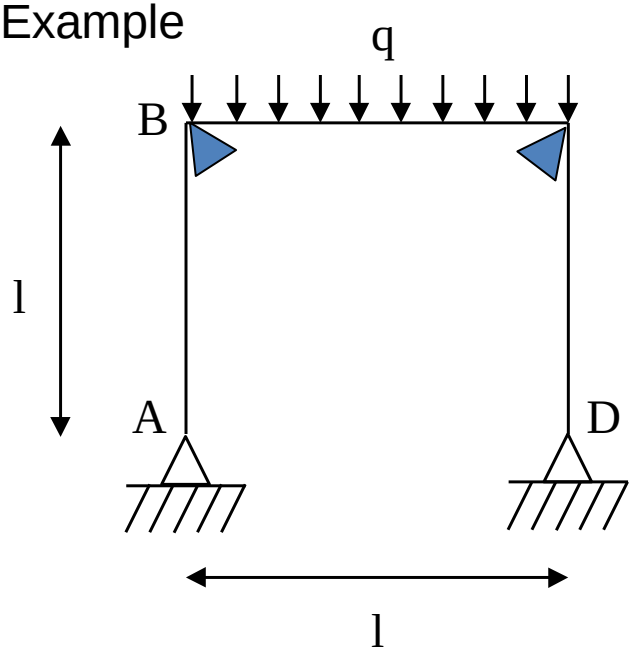
$$\theta_{B_2}(\text{clock wise}) = \frac{M_1 l}{3EI}$$

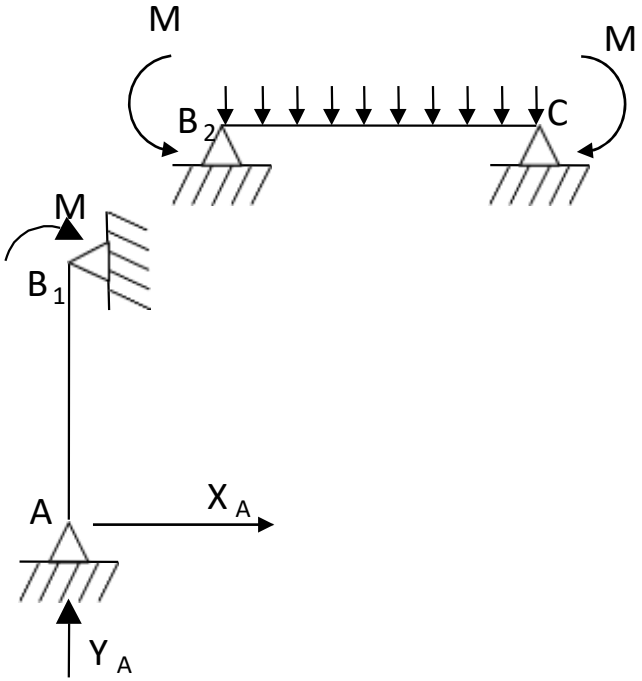
$$M_2 = \frac{2}{5} M \qquad M_1 = \frac{3}{5} M$$

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## ❑ Non-traslational frames

Example




$$\theta_{B_1}(\text{clock wise}) = \theta_{B_2}(\text{clock wise})$$

$$\theta_{B_1}(\text{clock wise}) = \frac{Ml}{3EI}$$

$$\theta_{B_2}(\text{clock wise}) = \frac{ql^3}{24EI} - \frac{Ml}{3EI} - \frac{Ml}{6EI}$$

$$M = \frac{ql^2}{20}$$

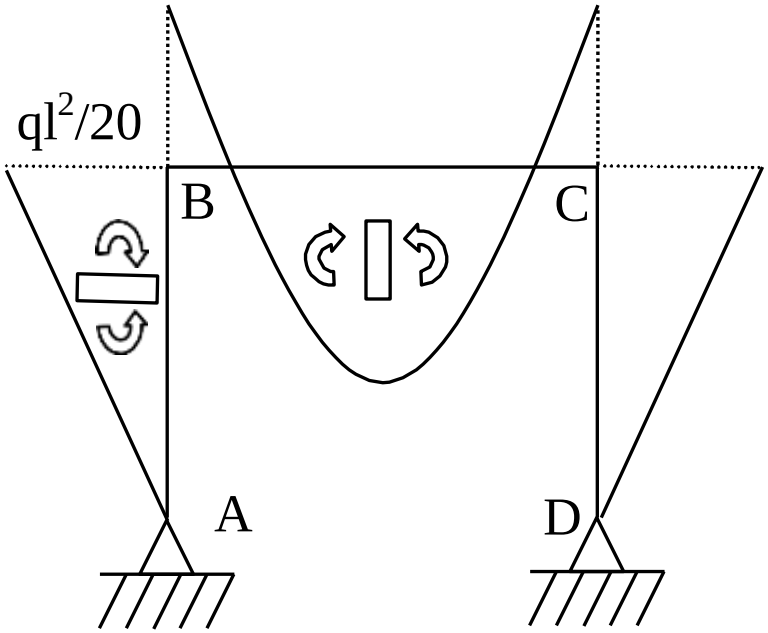
$$Y_A = \frac{ql}{2} \qquad X_A = \frac{ql}{20}$$

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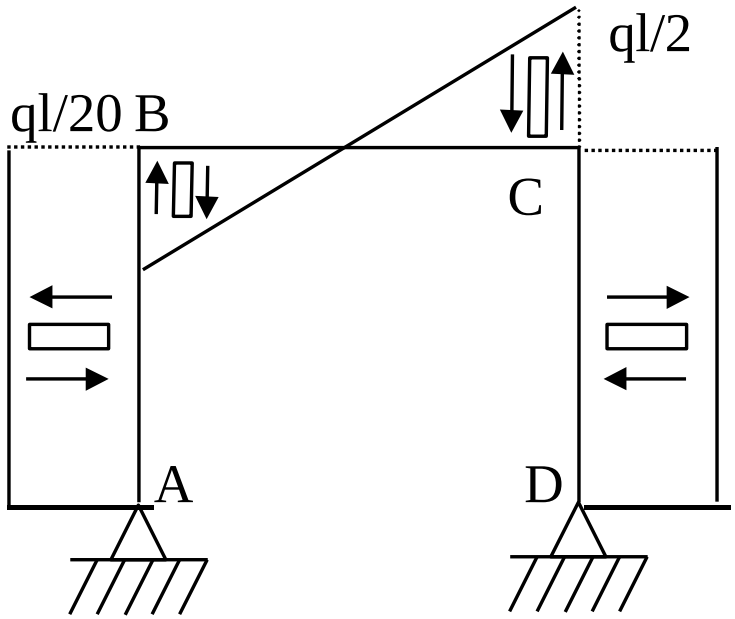
# Frame structures

## ❑ Non-traslational frames

Example

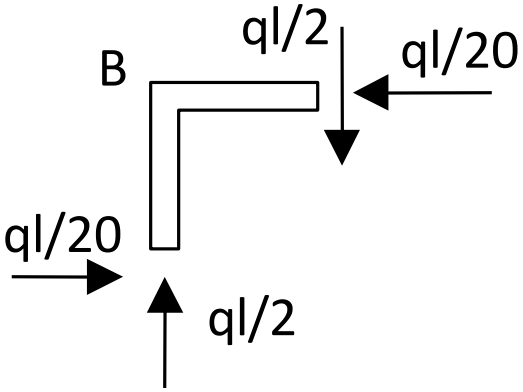



Bending moments law




Internal shear forces law

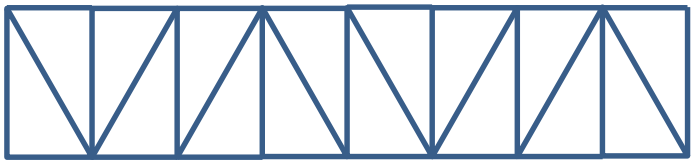
Node equilibrium!




Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Universidad Carlos III de Madrid

# Lattice structures






https://commons.wikimedia.org/wiki/Category:Hangars#/media/File:Hangaretes_do_Projeto_32_-_Foto_Set_2014.jpg

Nyttend



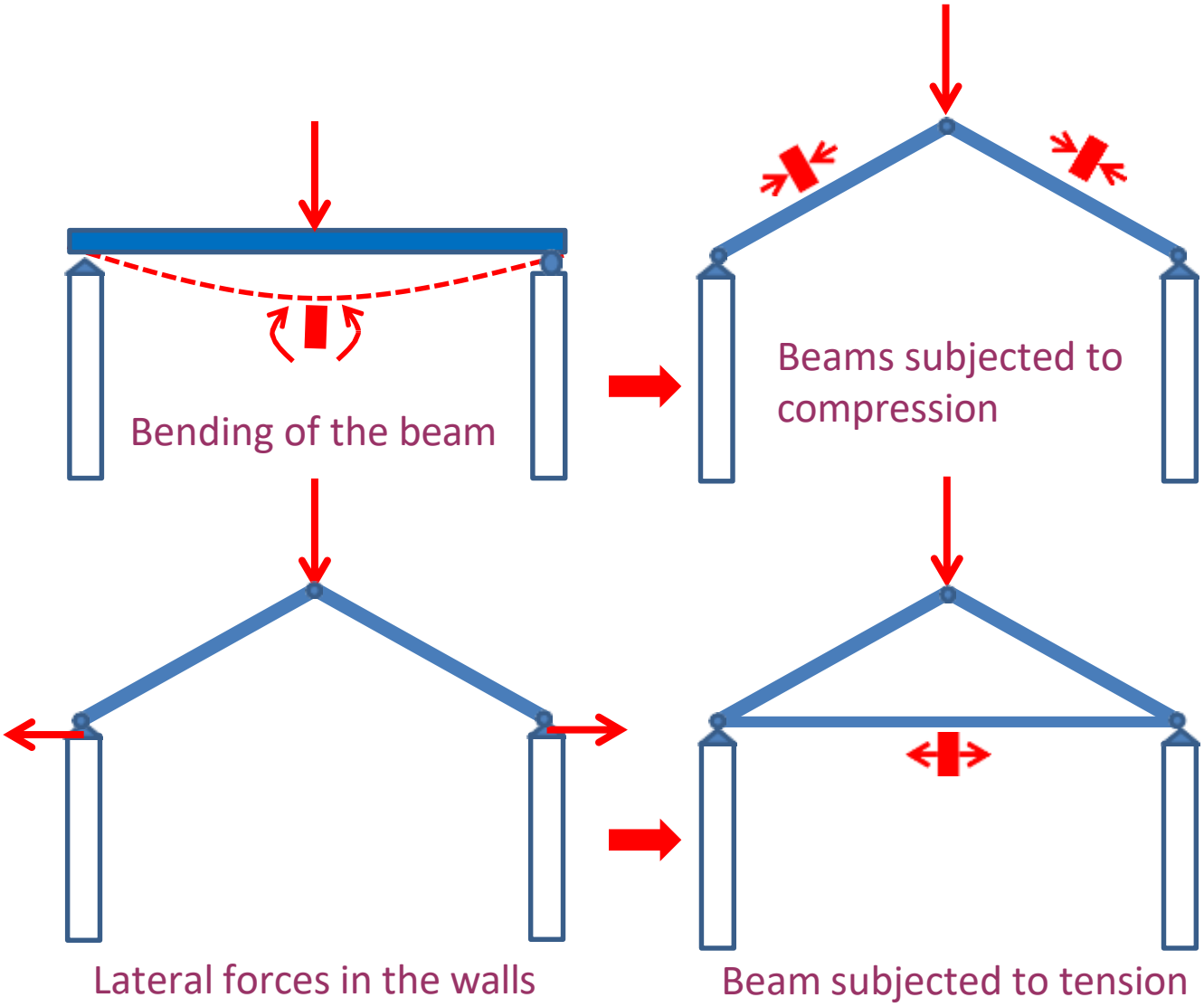
https://commons.wikimedia.org/wiki/File:Deep_Cut_Road_bridge_over_canal.jpg?uselang=es



De Uncredited. Original uploader was Mdnavman at en.wikipedia - http://www.flyingboats.ca/FlyingBoats-old/italy/italy.html - Transferred from en.wikipedia to Commons by User:EH101 using CommonsHelper., Dominio público, https://commons.wikimedia.org/w/index.php?curid=15112301

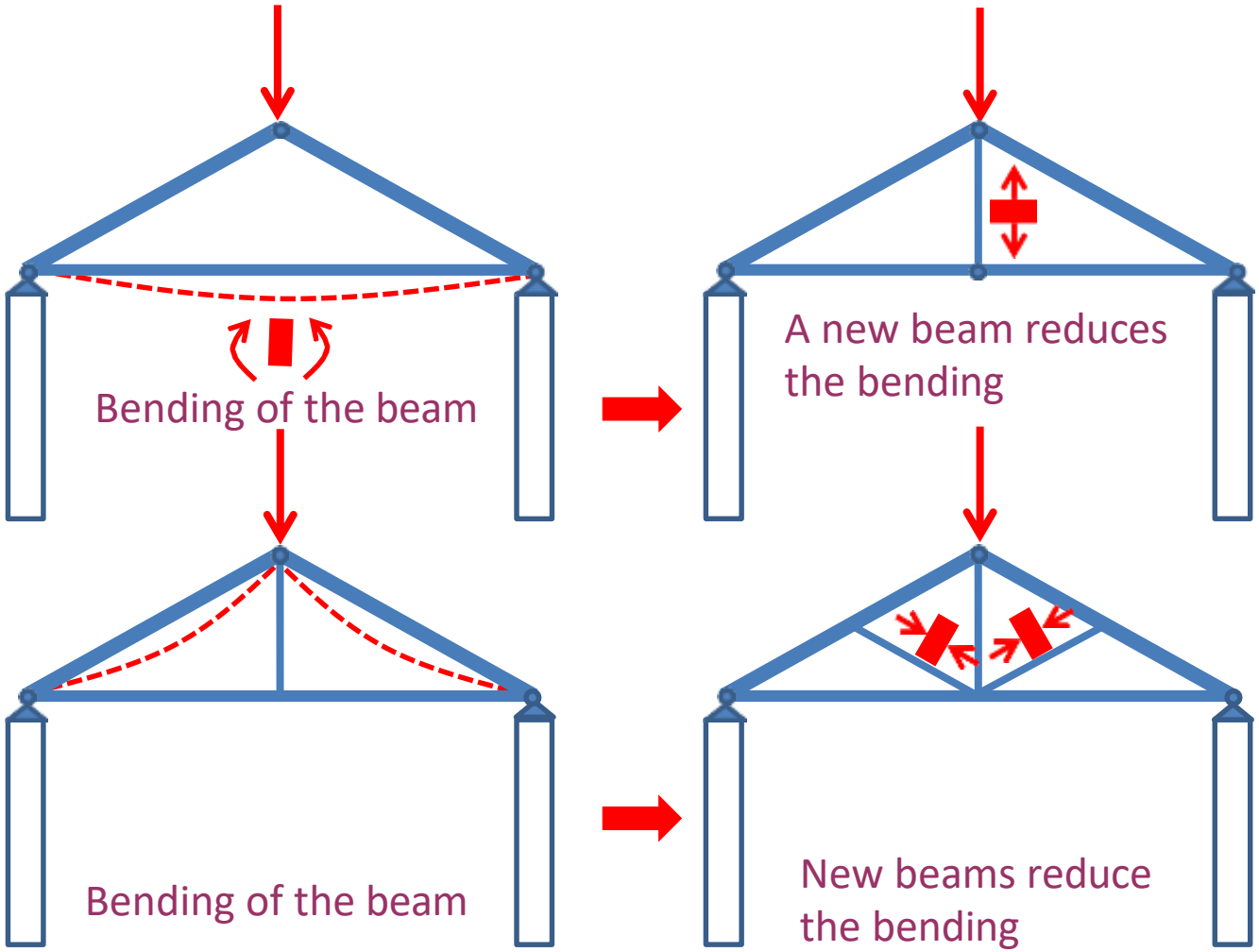
**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Lattice structures



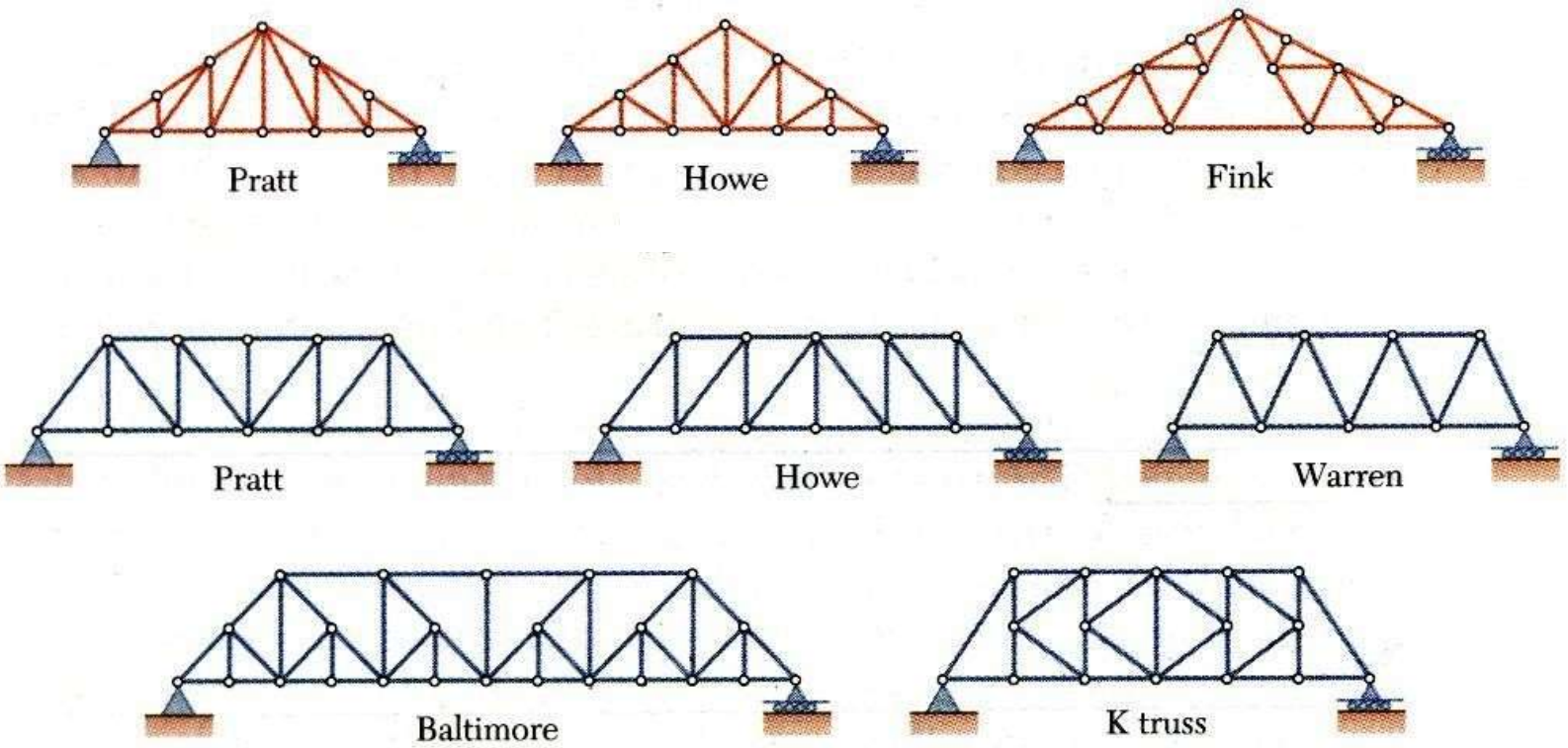

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Lattice structures




**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

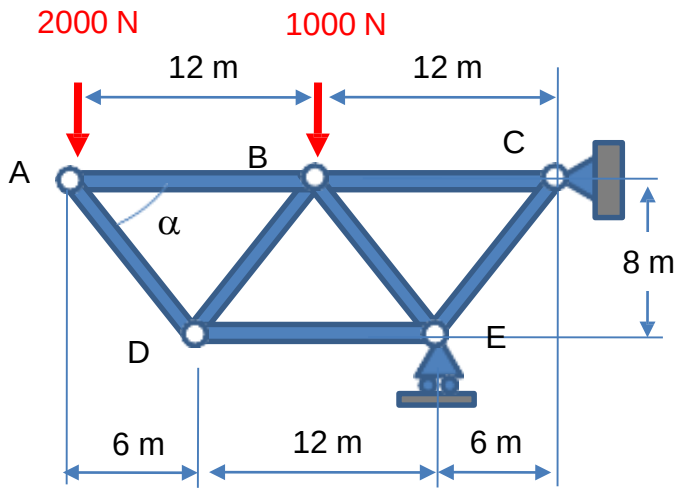
# Lattice structures




**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Lattice structures

Example




$$\sum M_C = 0 = 2000 \cdot 24 + 1000 \cdot 12 - V_E \cdot 6$$

$$V_E = 10\ kN \uparrow$$

$$\sum F_x = 0 = H_C \qquad H_C = 0$$

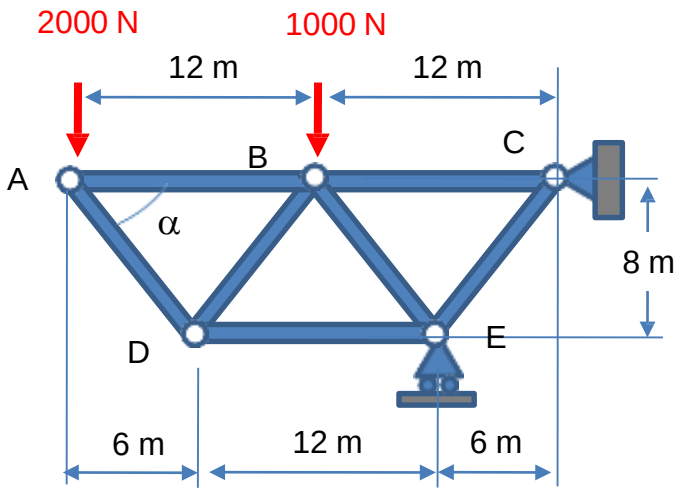
$$\sum F_y = 0 = -2000 - 1000 + V_E + V_C$$

$$V_C = 7\ kN \downarrow$$

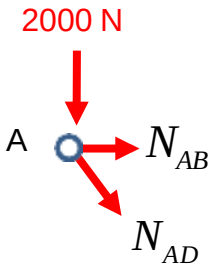
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Lattice structures

## Example




- Node A is subjected to only two unknown forces: applying equilibrium of forces in the node.




$$\sum F_y = 0 = 2000 + N_{AD} sen(\alpha)$$

$$\sum F_x = 0 = N_{AB} + N_{AD} \cos(\alpha)$$

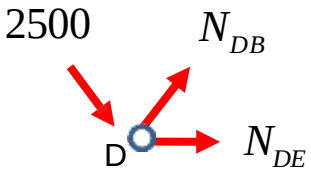
$$N_{AB} = 1500\ N \qquad N_{AD} = -2500\ \text{N}$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Lattice structures

Example

- Successively, the equilibrium of nodes D, B and E can be made.

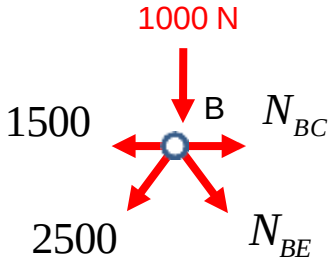



$$\sum F_y = 0 = -2500 sen(\alpha) + N_{DB} sen(\alpha)$$

$$\sum F_x = 0 = 2500 \cos(\alpha) + N_{DB} \cos(\alpha) N_{DB} + N_{DE}$$

$$N_{DB} = 2500 \ N$$

$$N_{DE} = -3000 \ N$$




$$\sum F_y = 0 = 1000 + (2500 + N_{BE}) sen(\alpha)$$

$$\sum F_x = 0 = N_{BC} + (N_{BE} - 2500) \cos(\alpha) - 1500$$

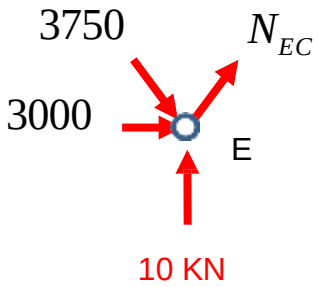
$$N_{BE} = -3750 \ N$$

$$N_{BC} = 5250 \ N$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Lattice structures

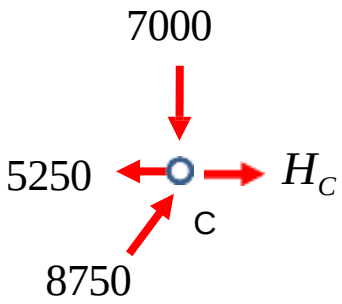
Example




$$\sum F_x = 0 = (N_{EC} + 3750)\cos(\alpha) + 3000$$

$$\boxed{N_{EC} = -8750 \text{ N}}$$

- All internal forces and reactions in C are known. The equilibrium in node C can be used to check the calculus.




$$\sum F_x = -5250 + 8750\cos(\alpha) = 0 \quad (\text{comprobado})$$

$$\sum F_y = -7000 + 8750 sen(\alpha) = 0 \quad (\text{comprobado})$$

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Bachelor in Aerospace Engineering



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# Aerospace Structures

## Chapter 2. Bending, shear and torsion of thin-walled beams

### Bending and shear of open and closed, thin-walled beams (I)

# Objetives

## CHAPTER 2. Bending, shear and torsion of thin-walled beams

## Bending and shear of open and closed, thin-walled beams (I)

**OBJECTIVES**

- State the displacement field the cross-section of a in a thin-walled beam
- Calculate normal stresses field in open and closed cross-sections
- Calculate normal stresses field in multi-cell cross-sections

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# Index

## CHAPTER 2. Bending, shear and torsion of thin-walled beams

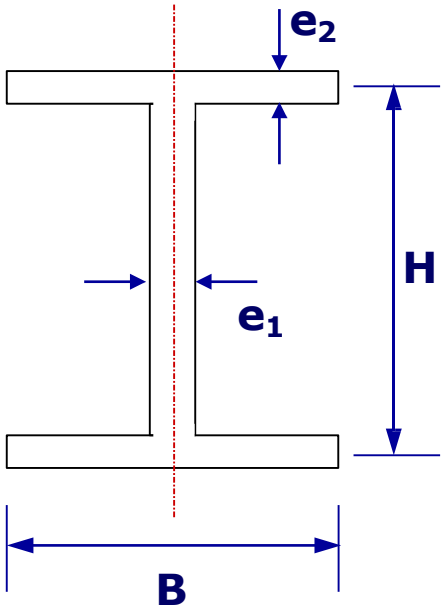
## Bending and shear of open and closed, thin-walled beams (I)


- Introduction
- Kinematics
- Normal stresses
- Shear stress field on open sections
- Stress field on single-cell closed sections
- Summary
- References

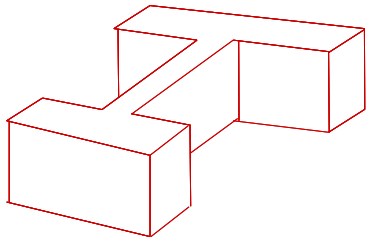

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# Introduction

## Cross-section of a thin-walled beam

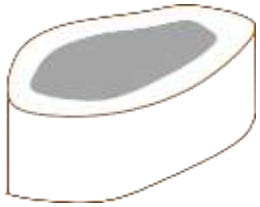



**OPEN**

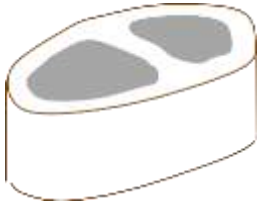


**CLOSED**

**Single cell**



**Multi cell**

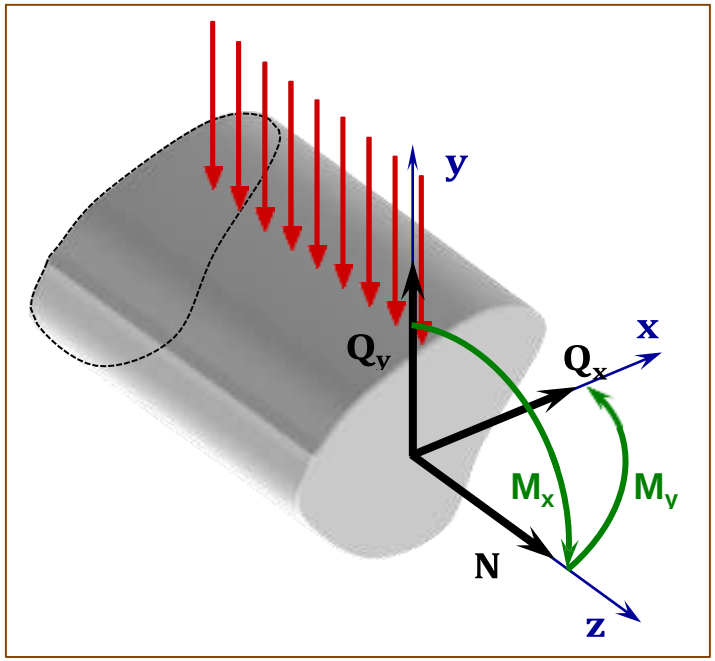


$$e_1 \ , \ e_2 \ \leq \frac{1}{10} \cdot \mathrm{Min}\{H, B\}$$

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## Previous knowledge (Introduction to Structural Analysis)




### Normal stresses

$$\sigma = \frac{N}{A} + \frac{y \cdot I_y - x \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} M_x + \frac{y \cdot P_{xy} - x \cdot I_x}{I_x \cdot I_y - P_{xy}^2} M_y$$

### Neutral axis

$$\frac{N}{A} + \frac{y \cdot I_y - x \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \cdot M_x + \frac{y \cdot P_{xy} - x \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \cdot M_y = 0$$

### Shear stresses

$$\tau_{yz} = \frac{Q_y}{a_o(y)} \cdot \left( \frac{m_{ex}^{A_1} \cdot I_y - m_{ey}^{A_1} \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \right) - \frac{Q_x}{a_o(y)} \cdot \left( \frac{m_{ex}^{A_1} \cdot P_{xy} - m_{ey}^{A_1} \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \right)$$

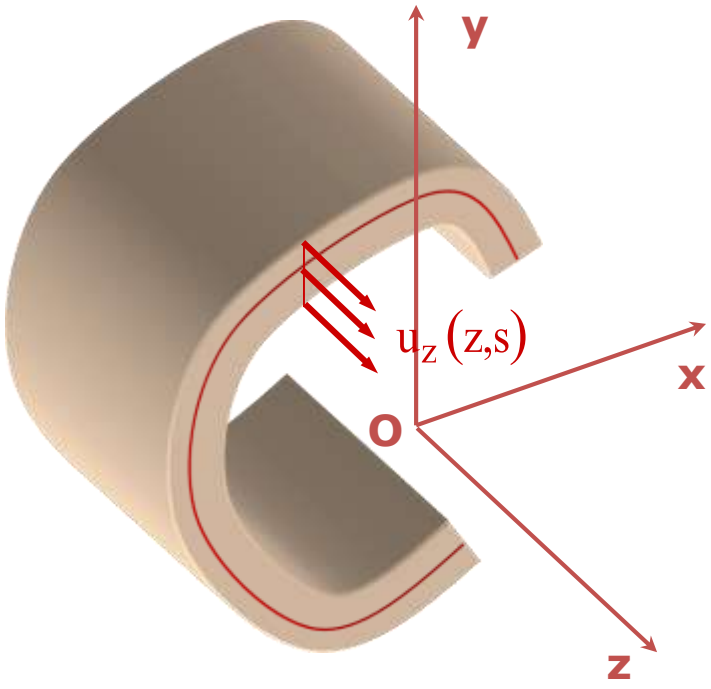
$$\tau_{xz} = \frac{Q_y}{b_o(x)} \cdot \left( \frac{m_{ex}^{A_2} \cdot I_y - m_{ey}^{A_2} \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \right) - \frac{Q_x}{b_o(x)} \cdot \left( \frac{m_{ex}^{A_2} \cdot P_{xy} - m_{ey}^{A_2} \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \right)$$

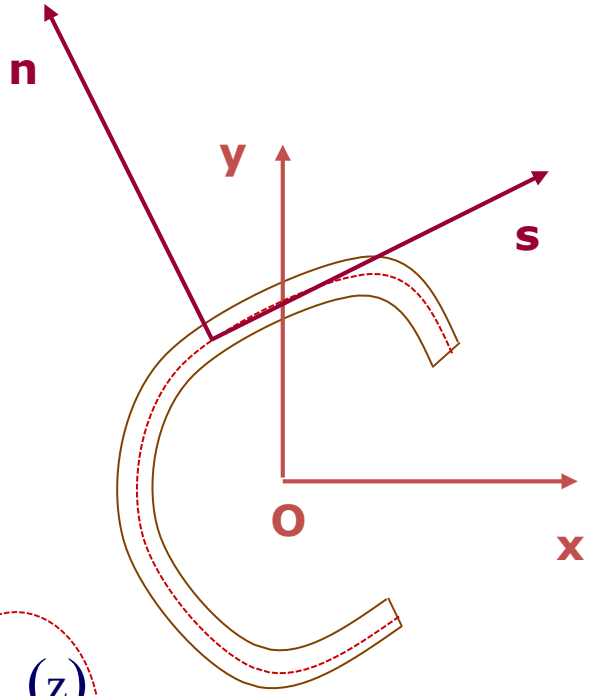
### Displacements field

$$u_z(x, y, z) = u_z(z) - \phi_y(z) \cdot x + \phi_x(z) \cdot y$$

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# Kinematics




$$u_x(x,y,z) = u_{x_o}(z)$$

$$u_y(x,y,z) = u_{y_o}(z)$$

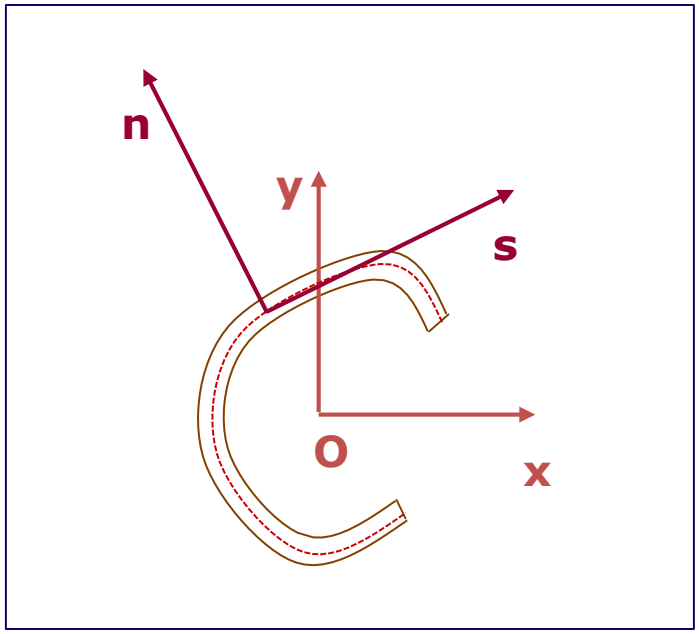
The section does not deform in its plane

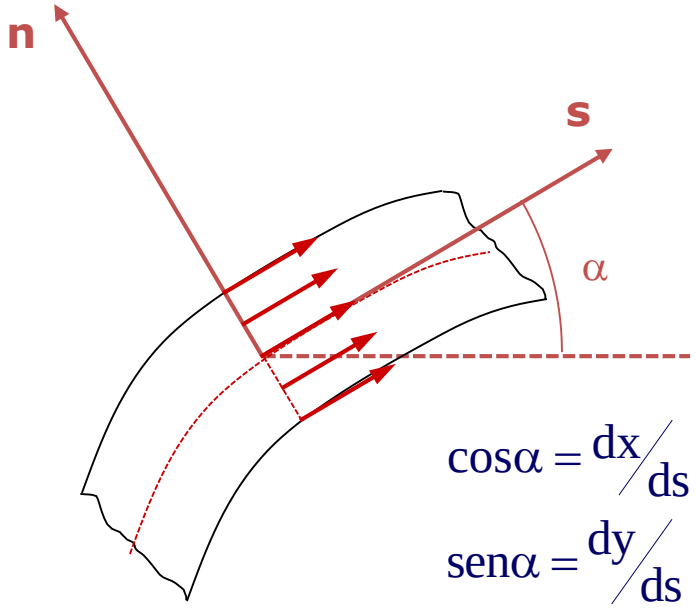
$$u_z(x,y,z) = u_z(z,s)$$

uz is constant along the thickness

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# Kinematics




$$\cos\alpha = dx/ds$$

$$\text{sen}\alpha = dy/ds$$

$$\gamma_{sz}(z,s) = \frac{\partial u_s(z,s)}{\partial z} + \frac{\partial u_z(z,s)}{\partial s}$$

$$u_s(z,s) = u_{x_o}(z)\cdot\cos\alpha + u_{y_o}(z)\cdot\text{sen}\alpha$$

$$u_z(z,s) = u_{z_o}(z) - \frac{du_{x_o}}{dz}\cdot x(s) - \frac{du_{y_o}}{dz}\cdot y(s) + \int_0^s \gamma_{sz}(z,s)\cdot ds$$

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# Normal stresses

$$u_z(z,s) = u_{z_o}(z) + \phi_x \cdot x(s) - \phi_y \cdot y(s)$$

$$\varepsilon_z(z,s) = \frac{\partial u_z(z,s)}{\partial z}$$

**Constitutive equation**

**Neglecting normal nonaxial stresses**

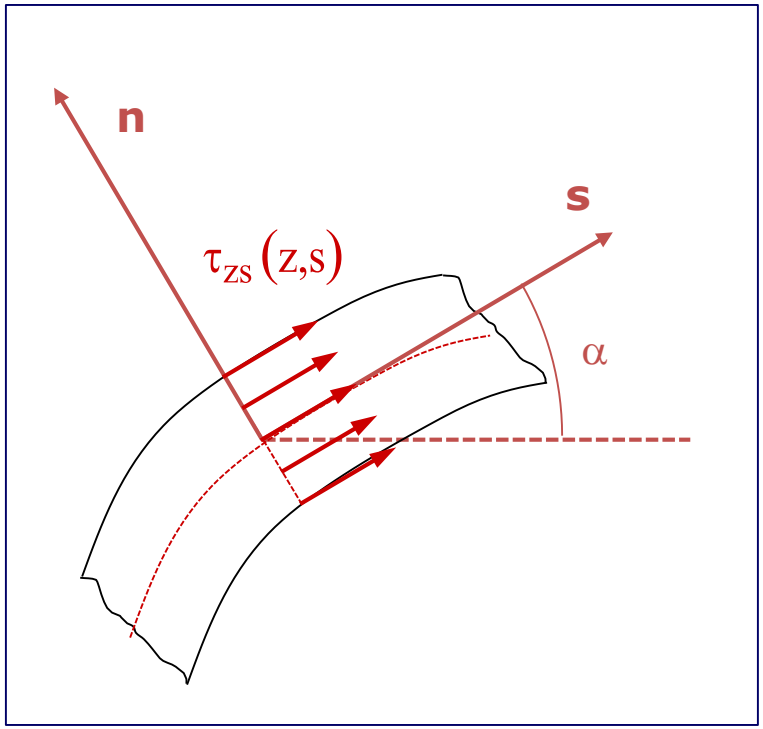
$$\sigma_z(z,s) = E \cdot \left( \frac{du_{z_o}(z)}{dz} + \frac{d\phi_x(z)}{dz} \cdot x(s) - \frac{d\phi_y(z)}{dz} \cdot y(s) \right)$$

**Operating as in solid sections**

$$\sigma_z(x,y,z) = \frac{N}{A} + \frac{y \cdot I_y - x \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \cdot M_x + \frac{y \cdot P_{xy} - x \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \cdot M_y$$

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# Shear stresses




$$\tau_{zs}(z,s) = cte$$

$$\tau_{sn}(z,s) = 0$$

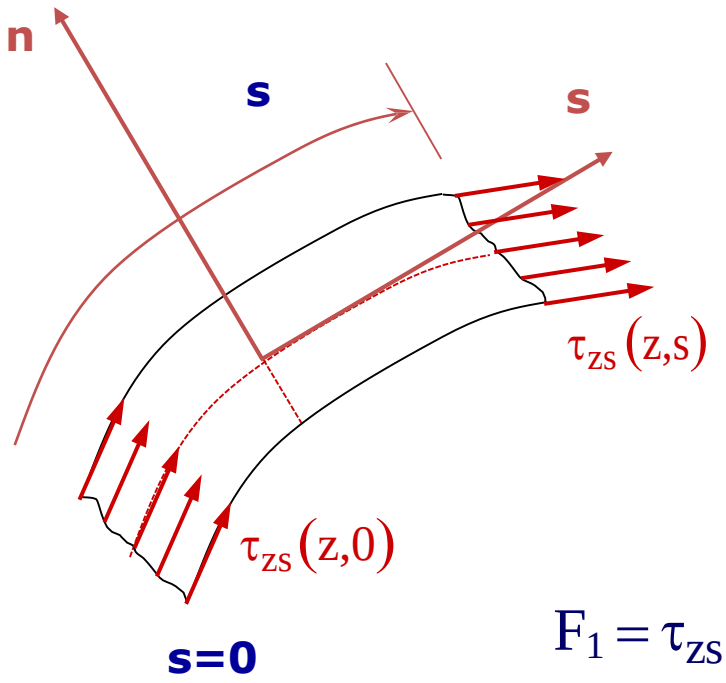
No deformations in cross-section

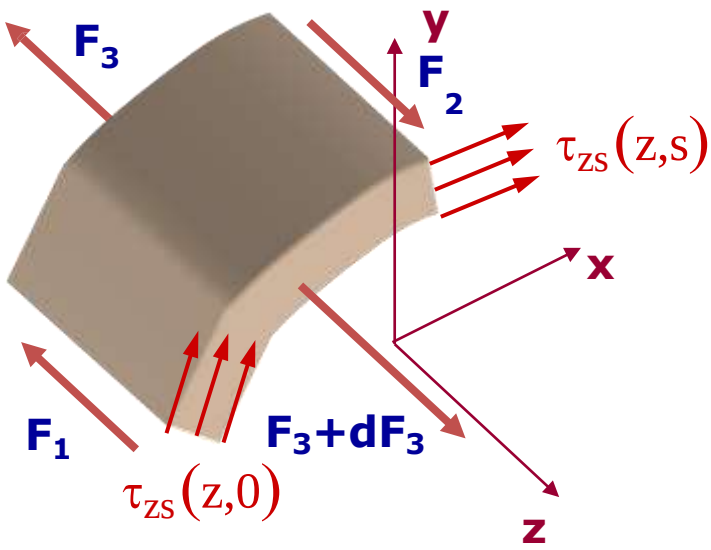
$$\tau_{zn}(z,s) \approx 0$$

Due to small thickness

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# Shear stresses



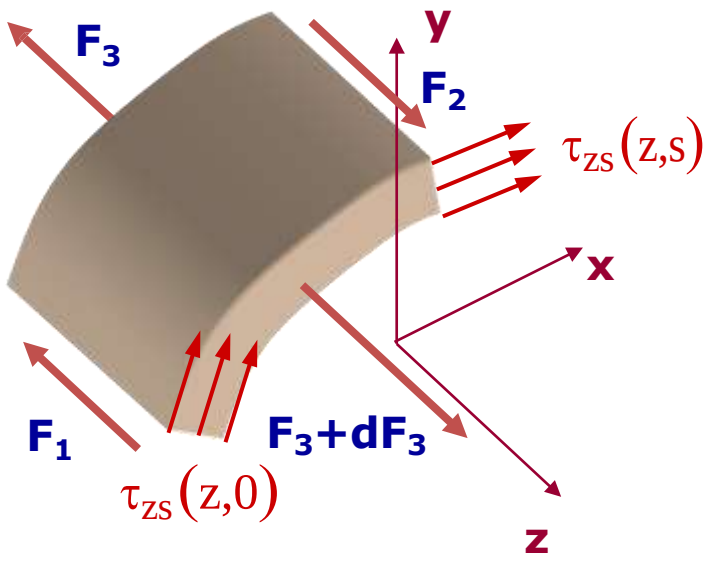

$$F_1 = \tau_{zs}(z,0)\cdot e(0)\cdot dz$$

$$F_2 = \tau_{zs}(z,s)\cdot e(s)\cdot dz$$

$$F_3 = \int_{A^*} \sigma_z(z,s)\cdot dA$$

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# Shear stresses




Equilibrium equations in z direction

$$F_3 + dF_3 - F_3 - F_1 + F_2 = 0$$

$$\tau_{zs}(z,s)\cdot e(s) = \tau_{zs}(z,0)\cdot e(0) - \frac{\partial}{\partial z}\int_{A^*}\sigma_z \cdot dA$$

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# Shear stress

$$\tau_{zs}(z,s)\cdot e(s) = \tau_{zs}(z,0)\cdot e(0) - \frac{\partial}{\partial z}\int_{A^*}\sigma_z \cdot dA$$

**Equilibrium equations**

**Espression of normal stresses**

$$\tau_{zs}(z,s)\cdot e(s) = \tau_{zs}(z,0)\cdot e(0) - K_y \cdot Q_y + K_x \cdot Q_x$$

**The distributed loads in z direction are neglected**

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Shear stresses

$$\tau_{zs}(z,s)\cdot e(s)=\tau_{zs}(z,0)\cdot e(0)-\frac{\partial}{\partial z}\int_{A^*}\sigma_z\cdot dA$$

$$\frac{dM_x}{dz}-Q_y+m_x=0 \qquad \frac{dQ_x}{dz}+q_x=0$$

$$\frac{dM_y}{dz}+Q_x+m_y=0 \qquad \frac{dQ_y}{dz}+q_y=0$$

$$\frac{dM_T}{dz}+m_z=0 \qquad \frac{dN}{dz}+q_z=0$$

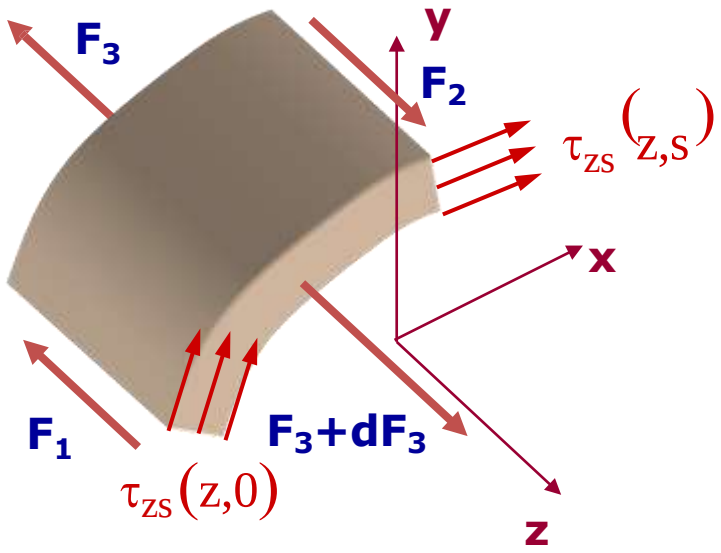
$$\sigma_z(x,y,z)=\frac{N}{A}+\frac{y\cdot I_y-x\cdot P_{xy}}{I_x\cdot I_y-P_{xy}^2}\cdot M_x+\frac{y\cdot P_{xy}-x\cdot I_x}{I_x\cdot I_y-P_{xy}^2}\cdot M_y$$

$$\tau_{zs}(z,s)\cdot e(s)=\tau_{zs}(z,0)\cdot e(0)-K_y\cdot Q_y+K_x\cdot Q_x$$

**The distributed loads in z direction are neglected**

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Shear stresses




$$K_y = \frac{m^*_{ex} \cdot I_y - m^*_{ey} \cdot P_{xy}}{I_x \cdot I_y - P^2_{xy}}$$

$$K_x = \frac{m^*_{ex} \cdot P_{xy} - m^*_{ey} \cdot I_x}{I_x \cdot I_y - P^2_{xy}}$$

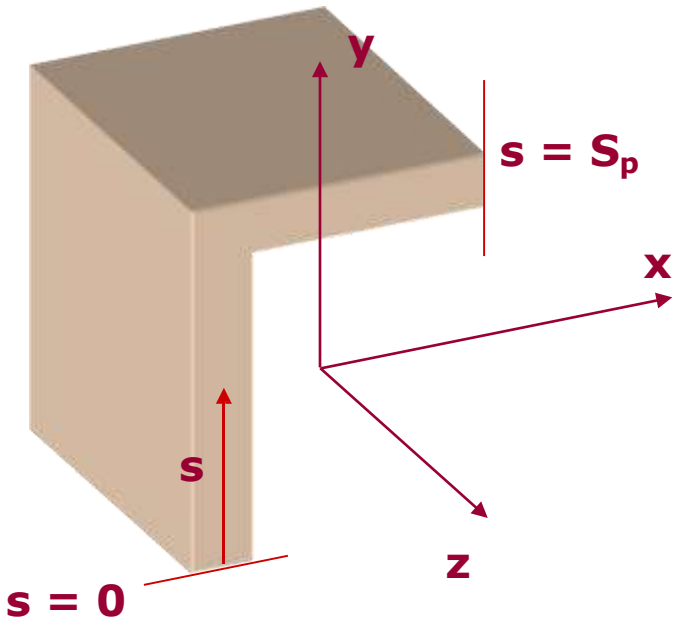
$$m^*_{ex} = \int_{A^*} y(s) \cdot dA = \int_0^s y(s) \cdot e(s) \cdot ds$$

$$m^*_{ey} = \int_{A^*} x(s) \cdot dA = \int_0^s x(s) \cdot e(s) \cdot ds$$

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# Shear stresses in open sections

## Open section




Assumption: no distributed loads in z direction

$$\tau_{zs}(z,0)\cdot e(0)=0$$

$$\tau_{zs}(z,s)\cdot e(s)=-K_{y}\cdot Q_{y}+K_{x}\cdot Q_{x}$$

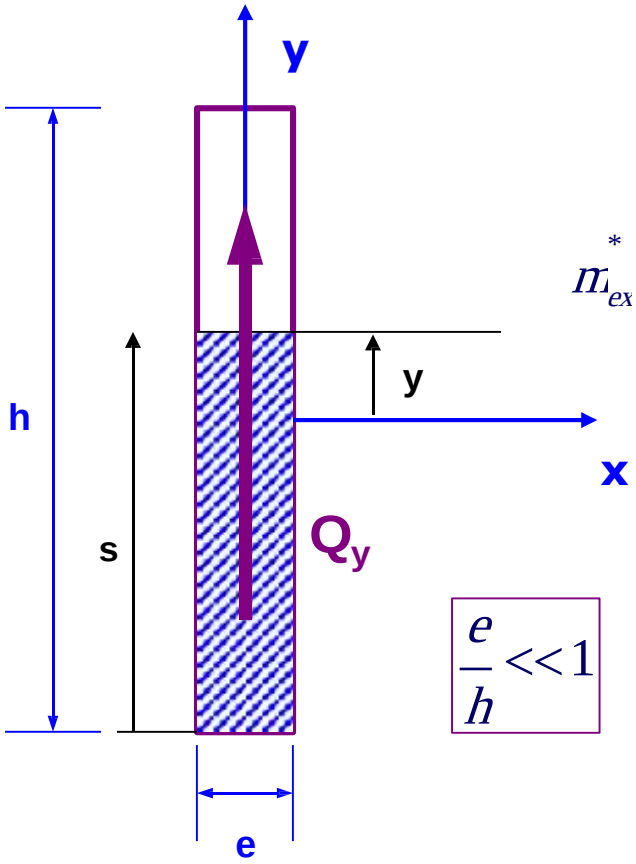
**Constant shear flow**

$$q(s)=\tau_{zs}(z,s)\cdot e(s)$$

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# Shear stresses in open sections

## Example: open section




$$\frac{e}{h} << 1$$

$$K_y = \frac{m^*_{ex}}{I_x} \qquad \tau_{zs}(z,s)\cdot e(s) = -K_y \cdot Q_y$$

$$m^*_{ex} = \int_0^s y(s)\cdot e(s)\cdot ds = \int_0^s \left(s - \frac{h}{2}\right)\cdot e\cdot ds = \frac{s\cdot e}{2}\cdot(s-h)$$

$$q(s) = -\frac{Q_y}{2\cdot I_x}\cdot s\cdot e\cdot(s-h)$$

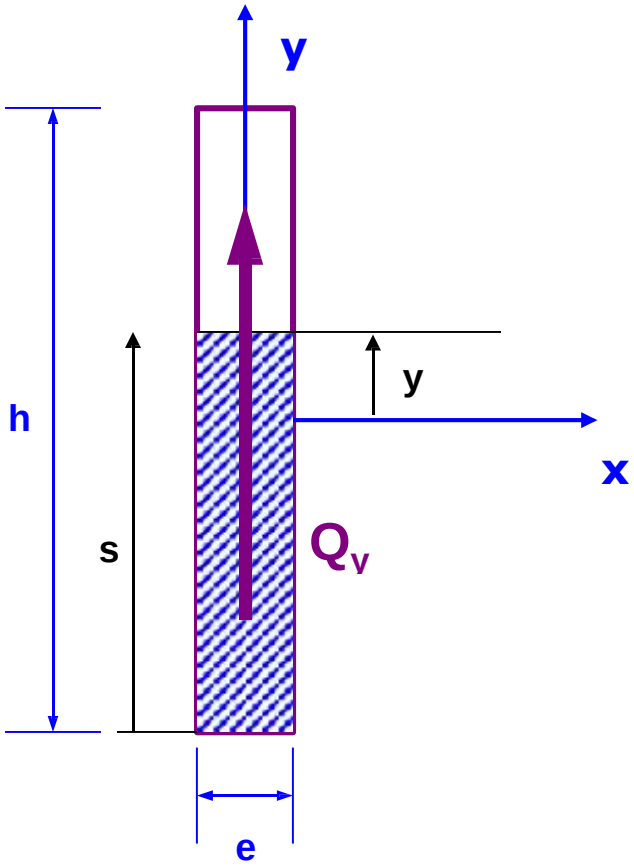
$$I_x = \frac{1}{12}\cdot e\cdot h^3$$

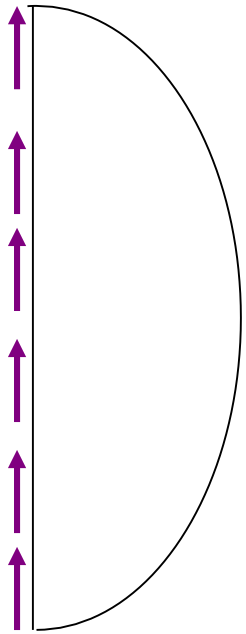
$$q(s) = -\frac{6\cdot Q_y}{h^3}\cdot s\cdot(s-h)$$

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# Shear stresses in open sections

## Example: open section



$$q(s) = \frac{6 \cdot Q_y}{h^3} \cdot s \cdot (h - s)$$

$$s = 0 \qquad q(0) = 0$$

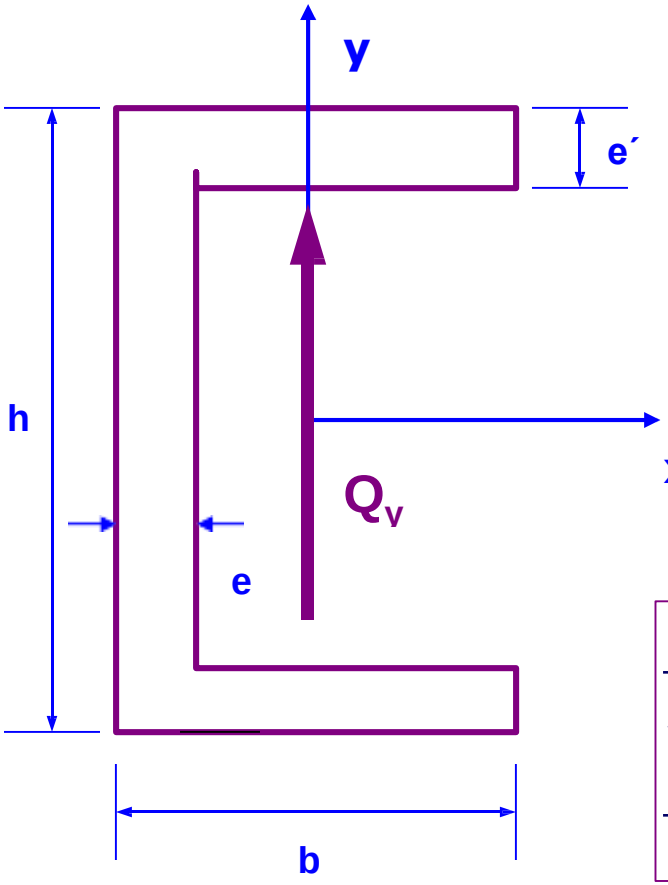
$$s = \frac{h}{2} \qquad q\left(\frac{h}{2}\right) = \frac{3}{2} \cdot \frac{Q_y}{h}$$

$$s = h \qquad q(h) = 0$$

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# Shear stresses in open sections

## Example: open section




$$\frac{e}{h} << 1$$

$$\frac{e'}{b} << 1$$

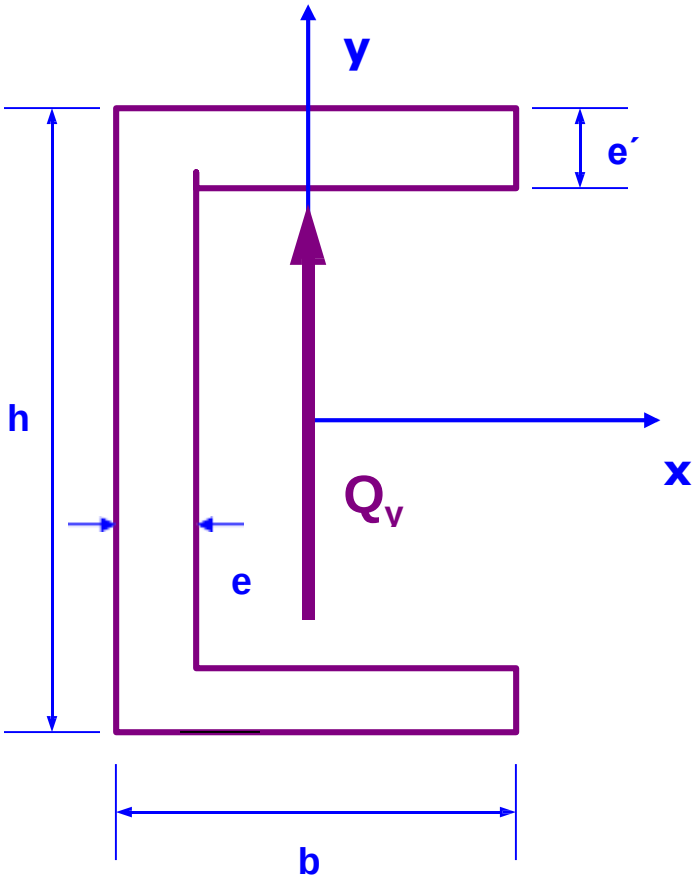
$$K_y = \frac{m^*_{ex}}{I_x}$$

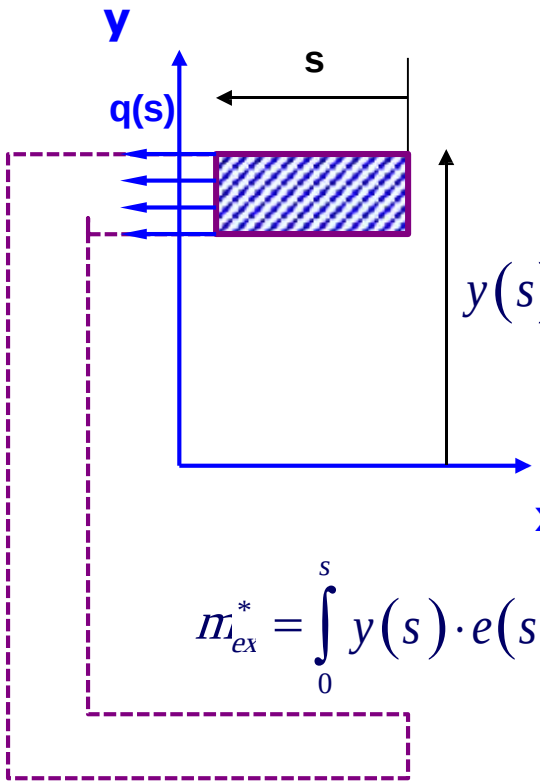
$$q(s) = -K_y \cdot Q_y$$

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# Shear stresses in open sections

## Example: open section




$$y(s) = \frac{h}{2}$$

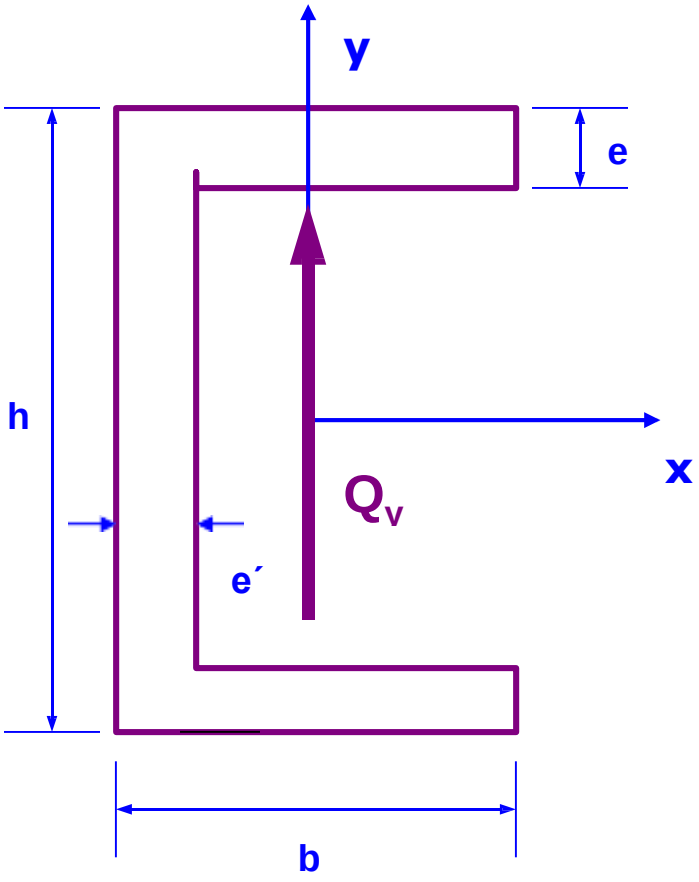
$$m_{ex}^{*} = \int_0^s y(s) \cdot e(s) \cdot ds = \frac{s \cdot e' \cdot h}{2}$$

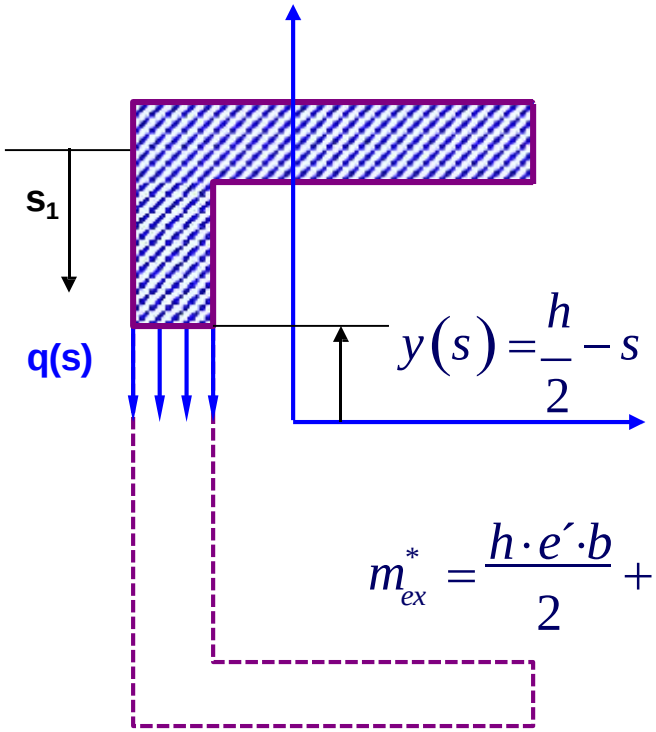
$$q(s) = -\frac{Q_y}{I_x} \cdot \frac{s \cdot e' \cdot h}{2}$$

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# Shear stresses in open sections

## Example: open section



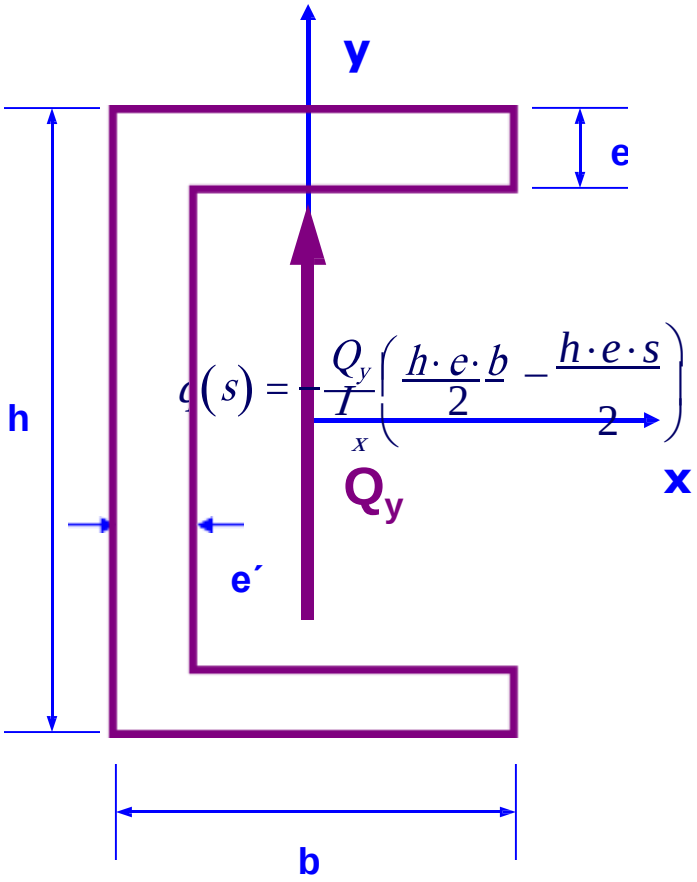

$$m_{ex}^{*} = \frac{h \cdot e' \cdot b}{2} + \frac{s_1 \cdot e}{2} \cdot (h - s_1)$$

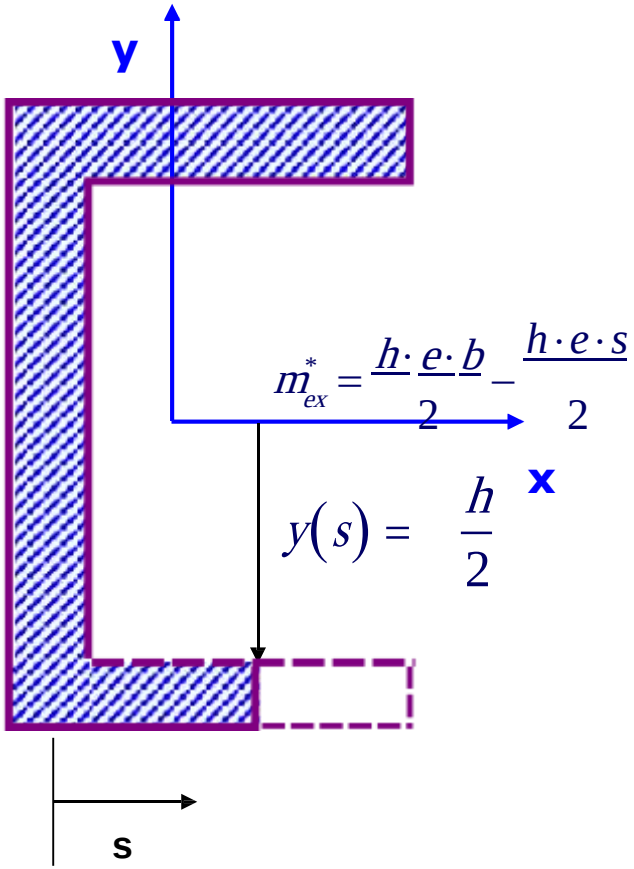
$$q(s) = -\frac{Q_y}{I_x} \cdot \left[ \frac{h \cdot e' \cdot b}{2} + \frac{s_1 \cdot e}{2} \cdot (h - s_1) \right]$$

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# Shear stresses in open sections

## Example: open section

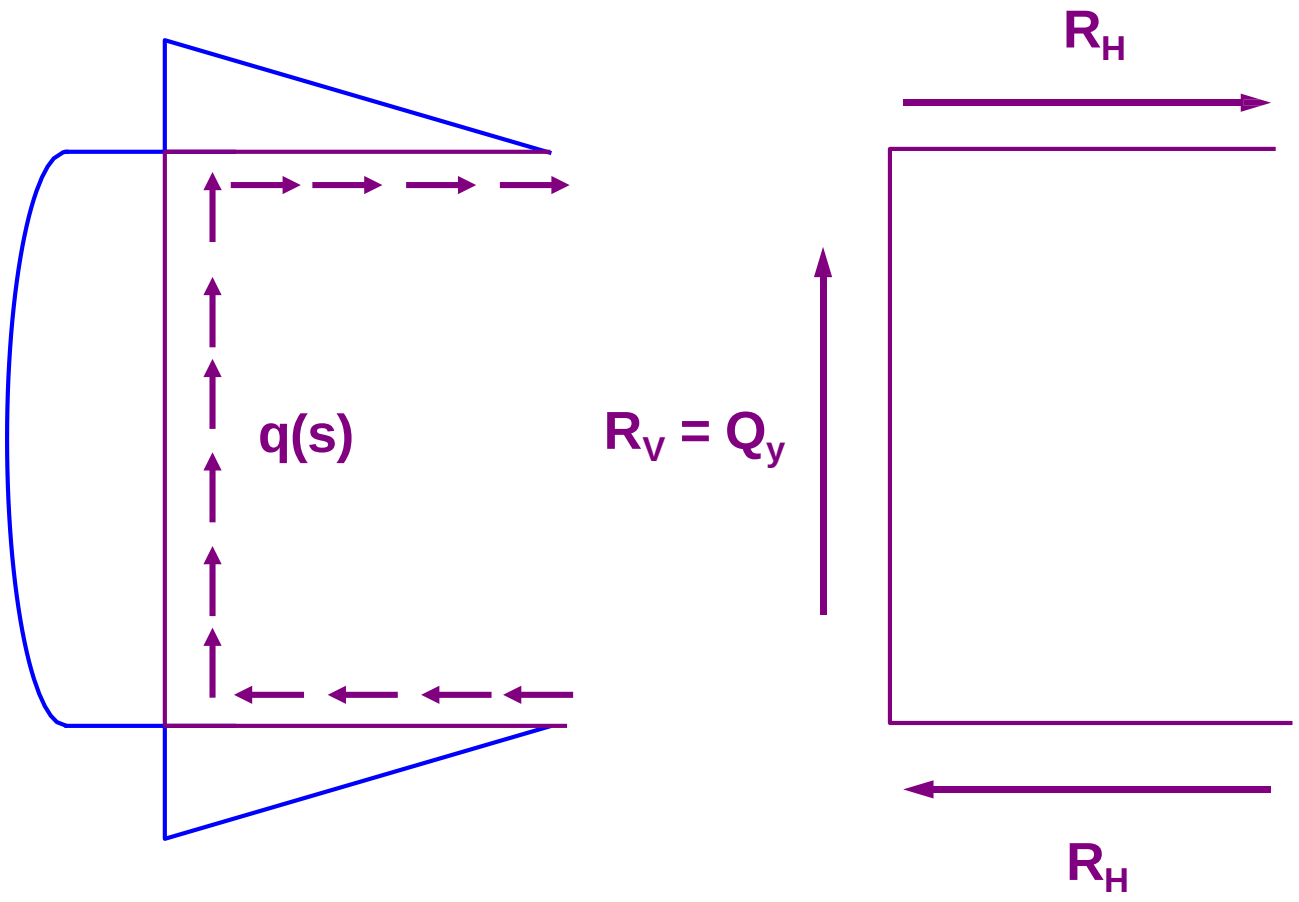



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# Shear stresses in open sections

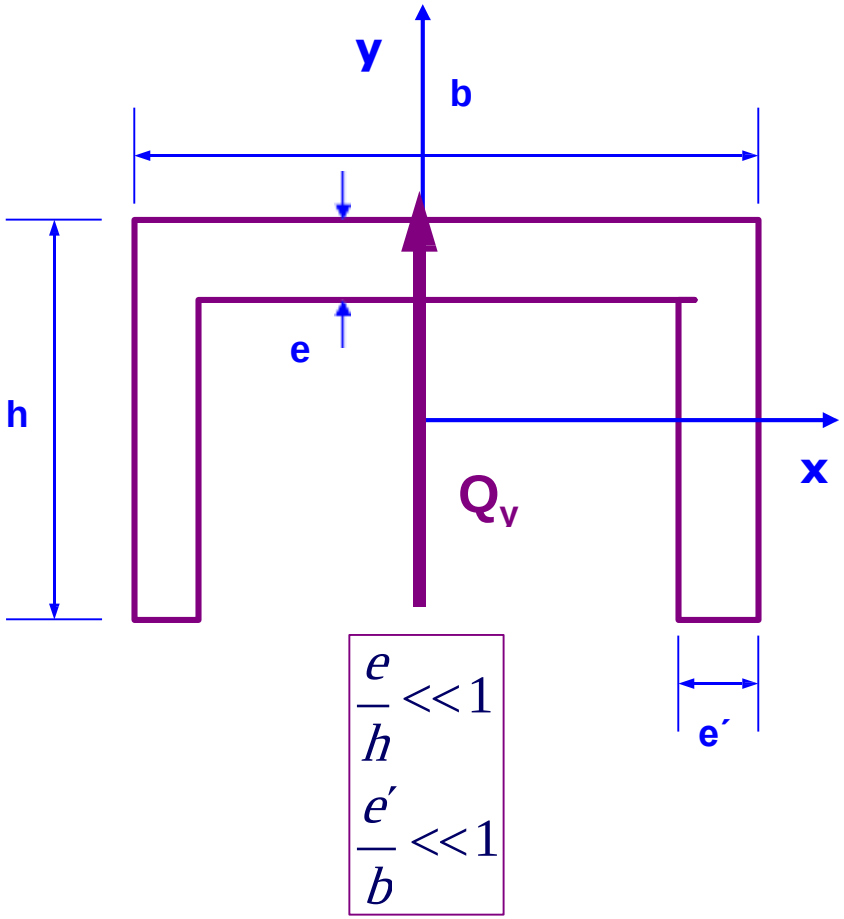
**Example: open section**



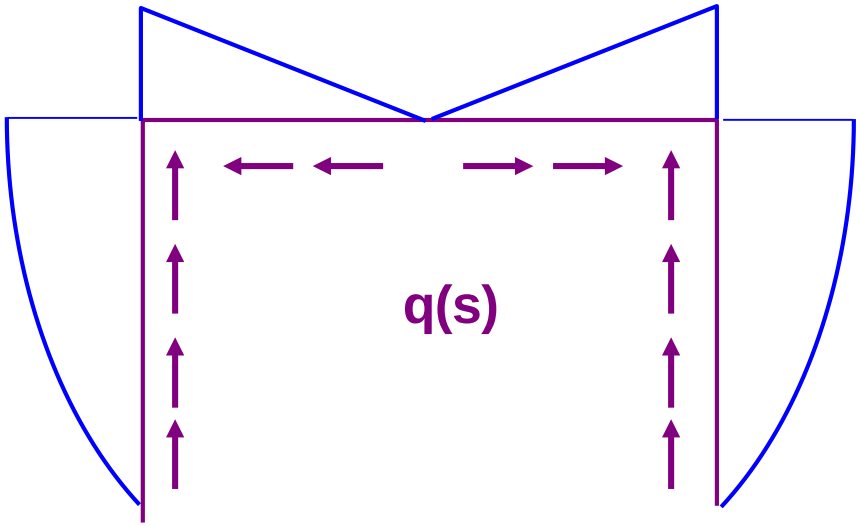

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# Shear stresses in open sections

**Example: open section**



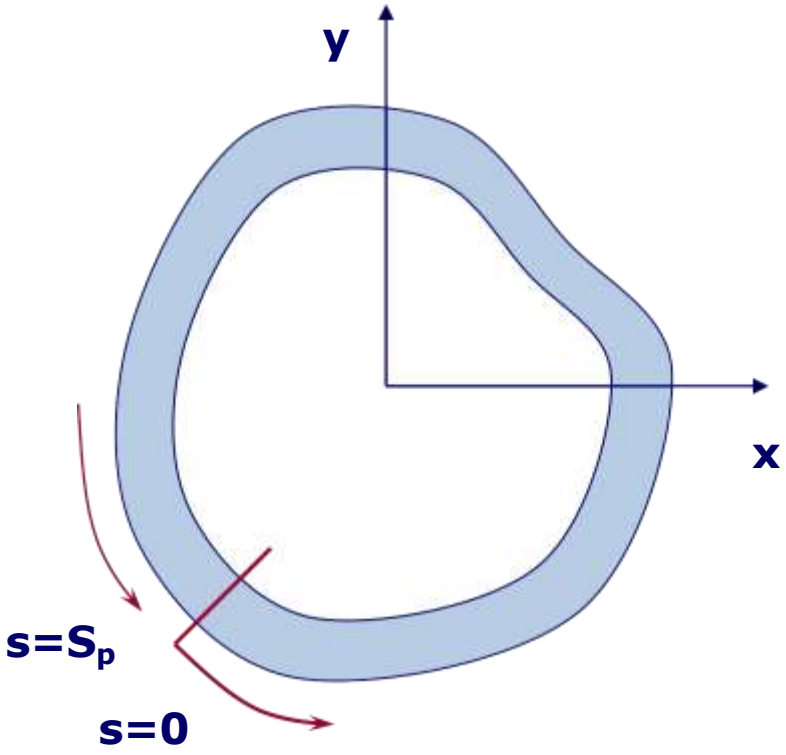

$$\frac{e}{h} << 1 \qquad \frac{e'}{b} << 1$$




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# Shear stresses in closed sections

## Single-cell closed section



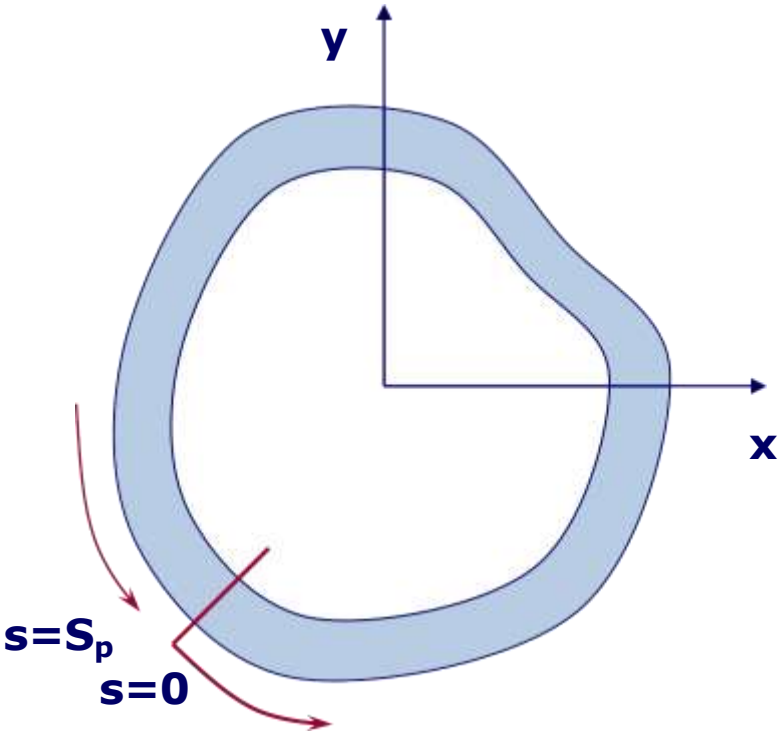

An arbitrary point is the origin of the coordinate *s*

$$q(s) = q(s)_{open} + q(0)$$

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# Shear stresses in closed sections

## Single-cell closed section




$$q(s) = q(s)_{open} + q(0)$$

$$\int_0^{S_p} \gamma_{zs} \cdot ds = 0$$

$$\int_0^{S_p} \frac{\tau_{zs}}{G} \cdot ds = 0$$

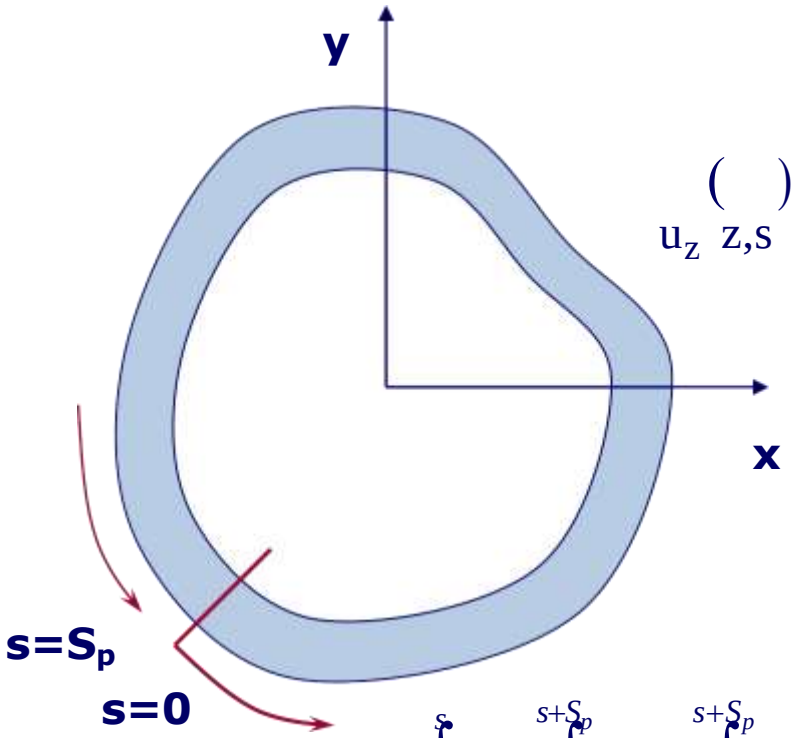
$$\int_0^{S_p} \frac{q(s)}{G \cdot e(s)}^{open} \cdot ds + \int_0^{S_p} \frac{q(0)}{G \cdot e(s)} \cdot ds = 0$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

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# Cálculo de tensiones tangenciales

## Single-cell closed section




$$u_z(z,s) = u_z(z,s+S_p)$$
$$y(s) = y(s+S_p)$$
$$z(s) = z(s+S_p)$$

$$u_z(z,s) = u_{z_o}(z) - \frac{du_{x_o}}{dz}\cdot x(s) - \frac{du_{y_o}}{dz}\cdot y(s) + \int_0^s \gamma_{sz}(z,s)\cdot ds$$

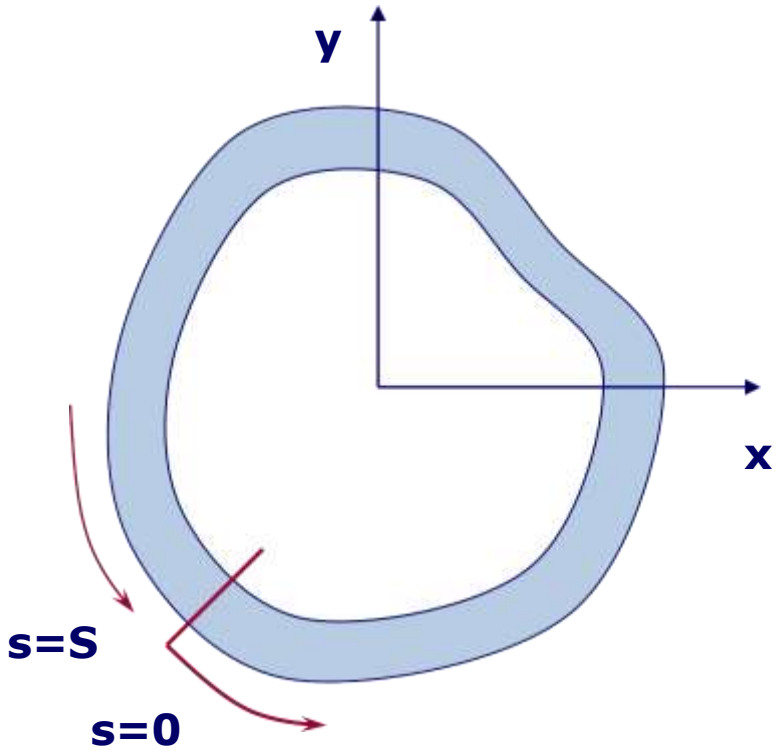
$$\int_0^s \gamma_{sz}(z,s)\cdot ds = \int_0^{s+S_p} \gamma_{sz}(z,s)\cdot ds$$

$$\int_0^s + \int_s^{s+S_p} = \int_0^{s+S_p} = \int_0^s \qquad \int_s^{s+S_p} = 0 \qquad \int_0^{S_p} \gamma_{sz}(z,s)\cdot ds = 0$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Shear stresses in closed sections

## Single-cell closed section



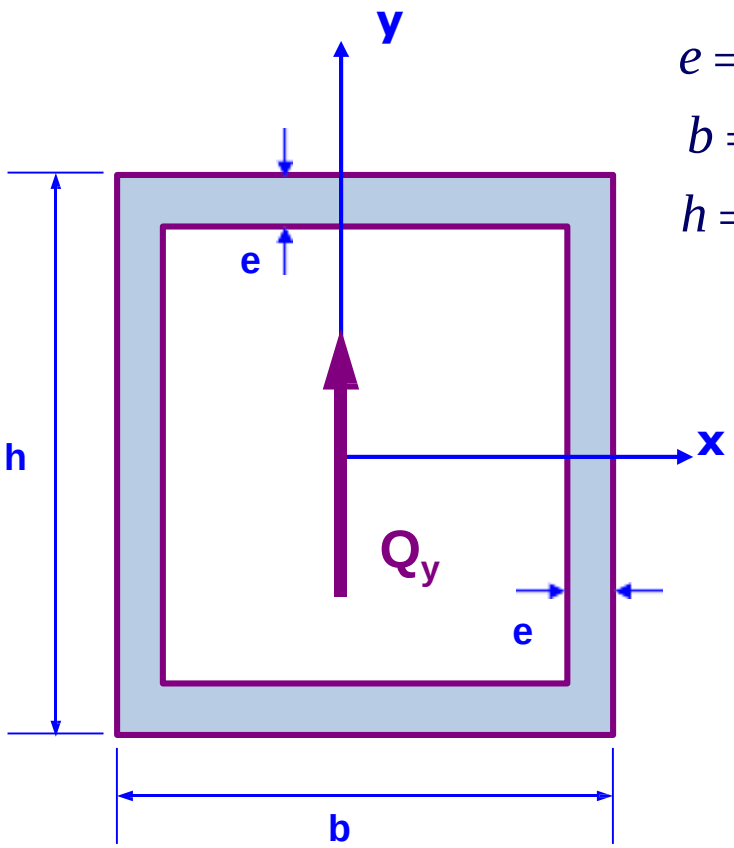

$$q(0) = -\frac{\int_0^S \frac{q(s)_{open}}{G \cdot e(s)} \cdot ds}{\int_0^S \frac{1}{G \cdot e(s)} \cdot ds}$$

$$q(s) = q(s)_{open} + q(0)$$

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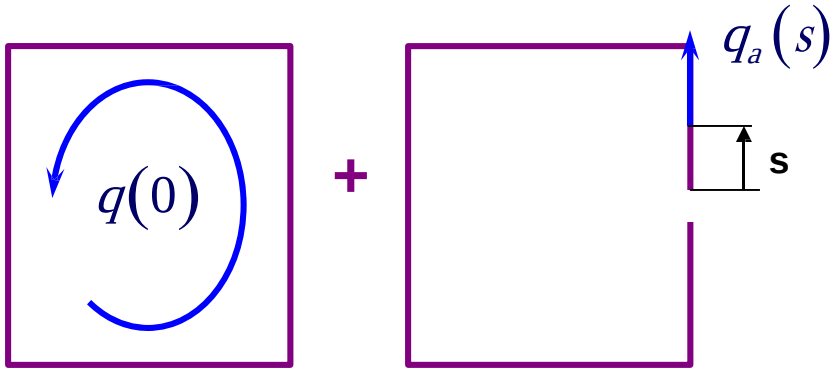
# Shear stresses in closed sections

## Single-cell closed section
## Example




$$e = 0{,}8 \ \ cm$$
$$b = 15 \ \ cm$$
$$h = 20 \ \ cm$$

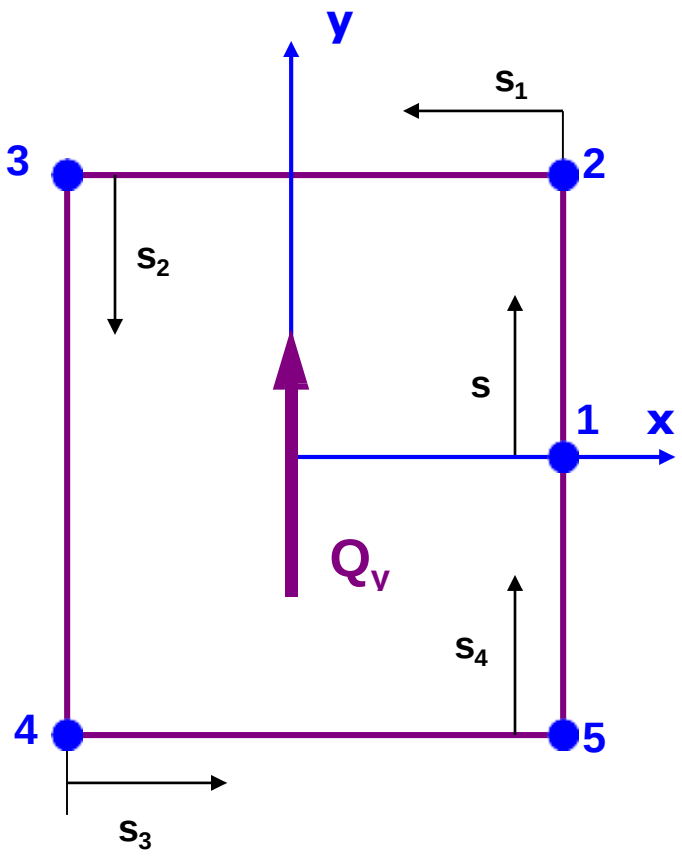
$$q(s) = q_a(s) + q(0)$$



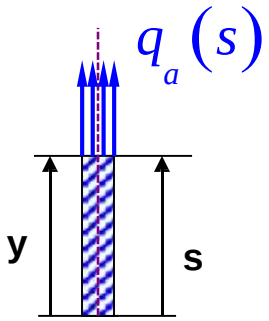

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# Shear stresses in closed sections

## Single-cell closed section
## Example




**zone 1-2**



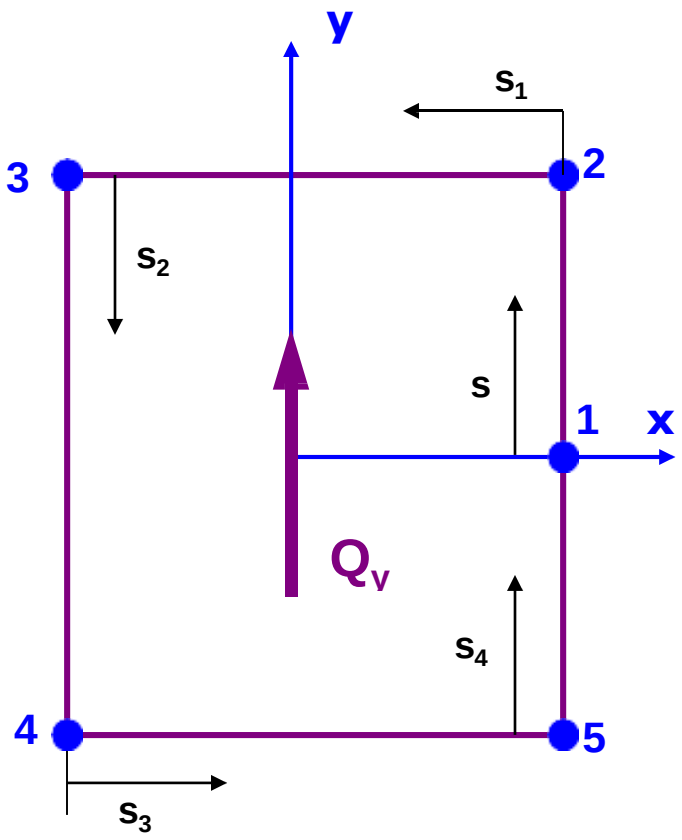

$$m_{ex}^{*} = \int_0^s y(s)\cdot e(s)\cdot ds = \frac{s^2\cdot e}{2}$$

$$q_a(s) = -\frac{Q_y}{I_x}\cdot\frac{e\cdot s^2}{2}$$

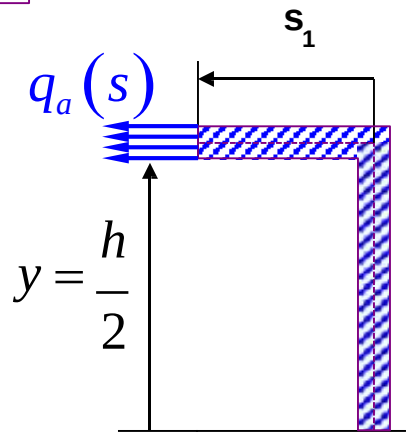
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# Shear stresses in closed sections

## Single-cell closed section
## Example




**zone 2-3**



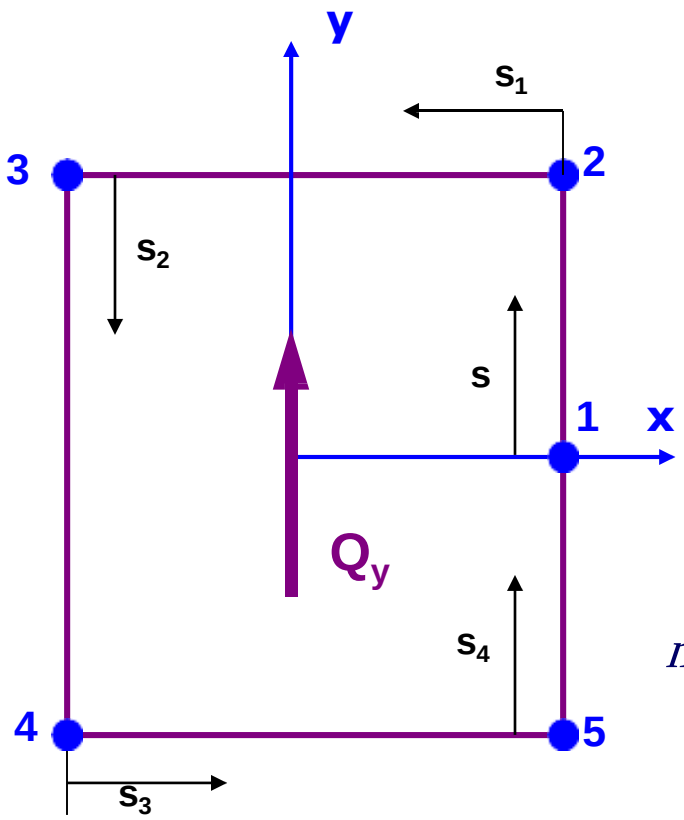

$$\pi^{*}_{ex} = \int_0^s y(s)\cdot e(s)\cdot ds = \frac{e\cdot h^2}{8} + \frac{h\cdot e\cdot s}{2}^{1}$$

$$q(s) = -\frac{Q_y}{I_x}\cdot\left(\frac{e\cdot h^2}{8} + \frac{h\cdot e\cdot s}{2}^{1}\right)$$

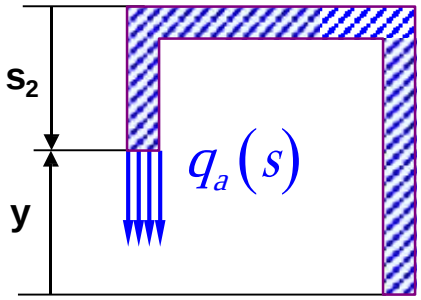
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# Shear stresses in closed sections

## Single-cell closed section
## Example




**zone 3-4**



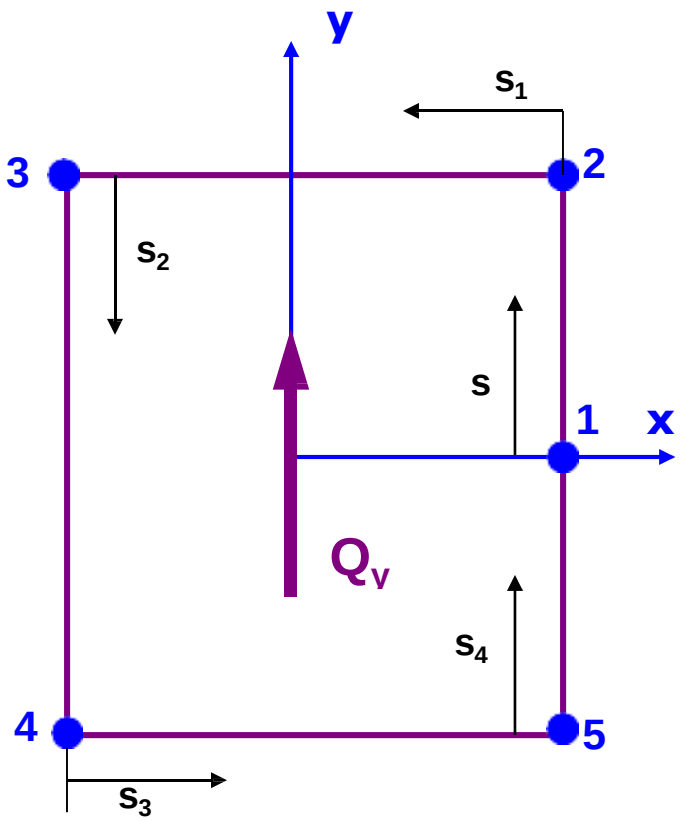

$$m_{ex}^{*} = \int_0^s y(s)\cdot e(s)\cdot ds = \frac{e\cdot h^2}{8} + \frac{h\cdot e\cdot b}{2} + \frac{e\cdot s^2}{2}\cdot(h - s_2)$$

$$q(s) = -\frac{Q_y}{I_x}\cdot\left[\frac{e\cdot h^2}{8} + \frac{h\cdot e\cdot b}{2} + \frac{e\cdot s^2}{2}\cdot(h - s_2)\right]$$

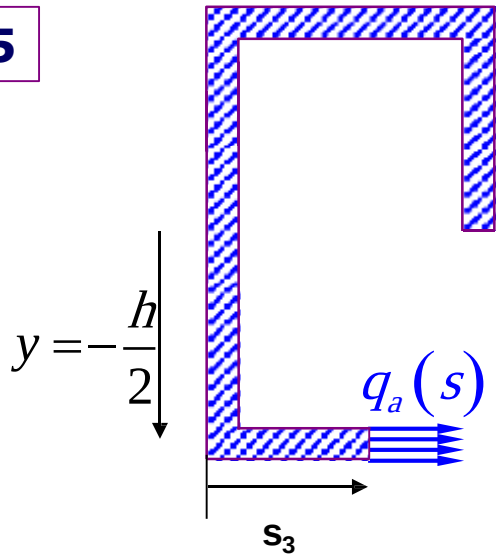
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# Shear stresses in closed sections

## Single-cell closed section
## Example




**zone 4-5**



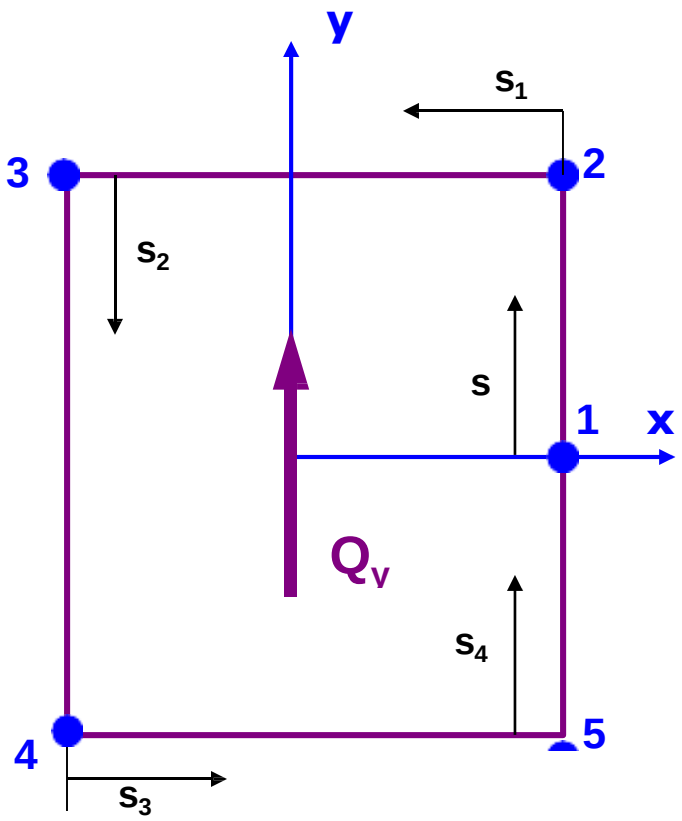

$$m^*_{ex} = \int_0^s y(s)\cdot e(s)\cdot ds = \frac{e\cdot h^2}{8} + \frac{h\cdot e\cdot b}{2} - \frac{h\cdot e}{2}\cdot s_3$$

$$q(s) = -\frac{Q_y}{I_x}\cdot\left[\frac{e\cdot h^2}{8} + \frac{h\cdot e\cdot b}{2} - \frac{h\cdot e}{2}\cdot s_3\right]$$

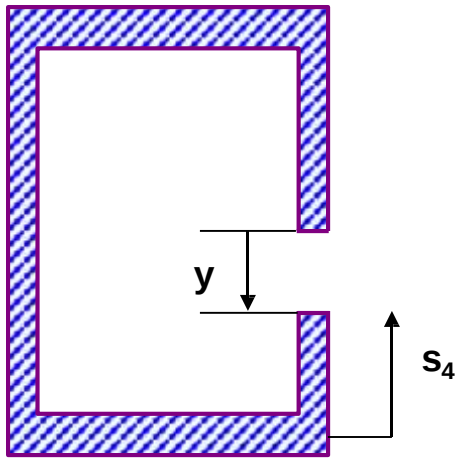
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# Shear stresses in closed sections

## Single-cell closed section
## Example




**zone 5-1**



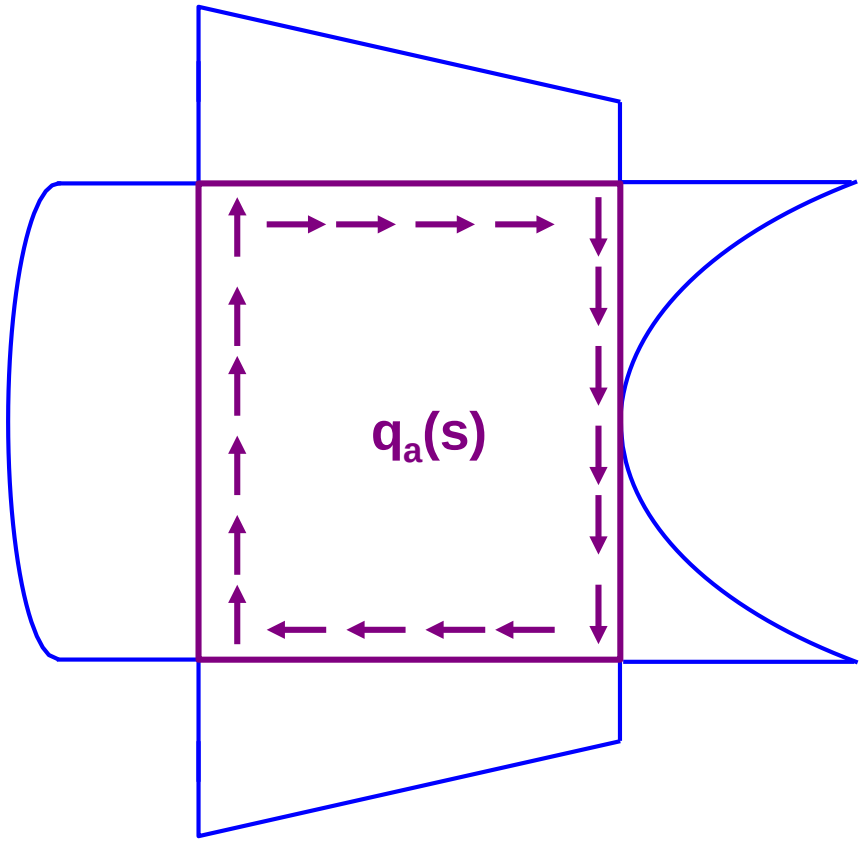

$$m^*_{ex} = \int_0^s y(s)\cdot e(s)\cdot ds = \frac{e\cdot h^2}{8} + + \frac{s_4\cdot e}{2}\cdot(s_4 - h)$$

$$q(s) = -\frac{Q_y}{I_x}\cdot\left[\frac{e\cdot h^2}{8} + \frac{s_4\cdot e}{2}\cdot(s_4 - h)\right]$$

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# Shear stresses in closed sections

## Single-cell closed section
## Example




$$q(0) = -\frac{\int_0^{2\cdot(h+b)} q_a(s)\cdot ds}{2\cdot(h+b)}$$

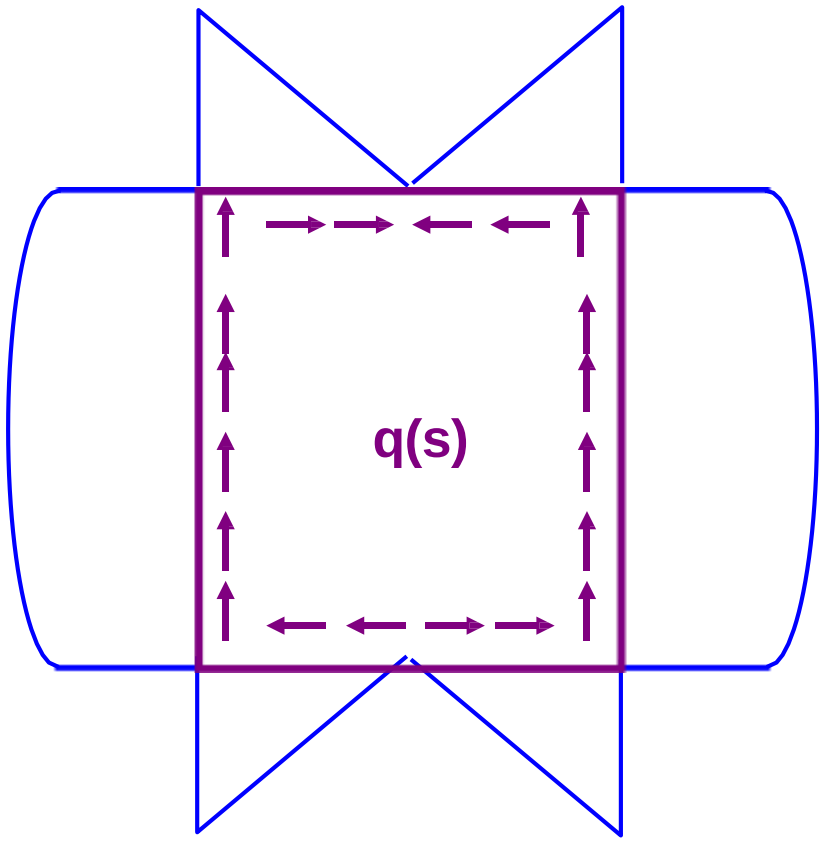
$$q(0) = 100\cdot\frac{Q_y}{I_x}$$

$$q(s) = q_a(s) + 100\cdot\frac{Q_y}{I_x}$$

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# Shear stresses in closed sections

## Single-cell closed section
## Example




$$q(0) = -\frac{\int_0^{2\cdot(h+b)} q_a(s)\cdot ds}{2\cdot(h+b)}$$

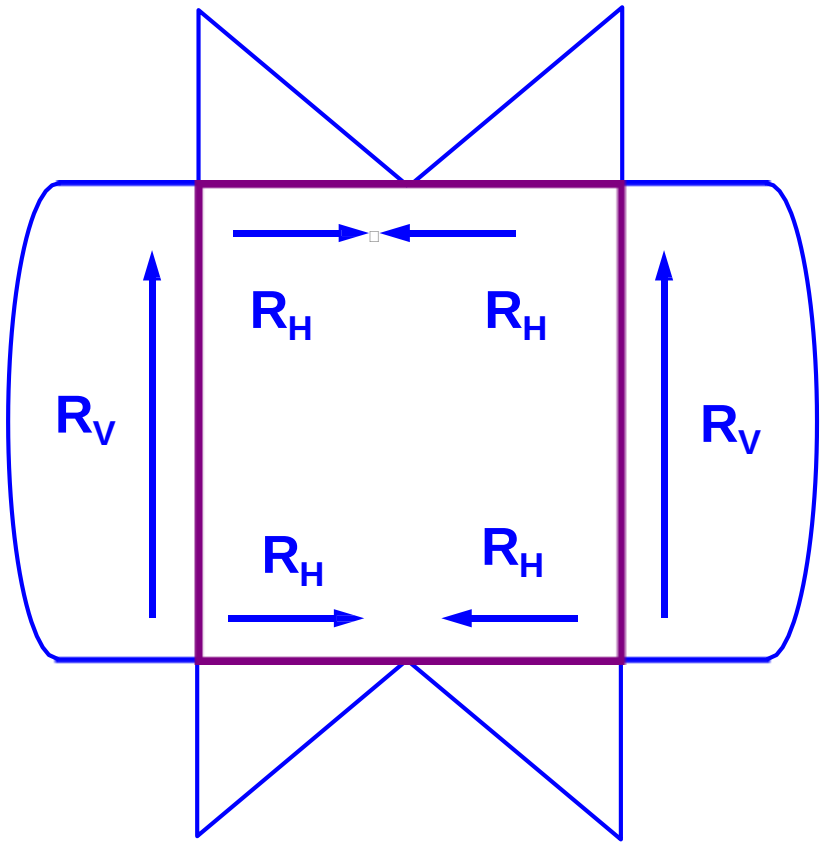
$$q(0) = 100\cdot\frac{Q_y}{I_x}$$

$$q(s) = q_a(s) + 100\cdot\frac{Q_y}{I_x}$$

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# Shear stresses in closed sections

## Single-cell closed section
## Example




$$q(0) = -\frac{\int_0^{2\cdot(h+b)} q_a(s)\cdot ds}{2\cdot(h+b)}$$

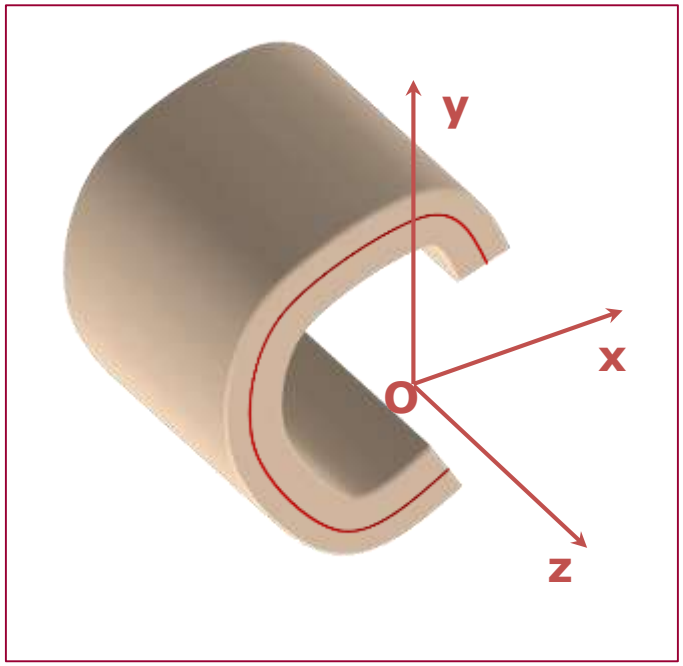
$$q(0) = 100\cdot\frac{Q_y}{I_x}$$

$$q(s) = q_a(s) + 100\cdot\frac{Q_y}{I_x}$$

$$R_V = \frac{Q_y}{2}$$

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# Summary




## Displacements

$$u_x(x,y,z) = u_{x_o}(z)$$

$$u_y(x,y,z) = u_{y_o}(z)$$

$$u_z(x,y,z) = u_z(z,s)$$

## Normal stresses

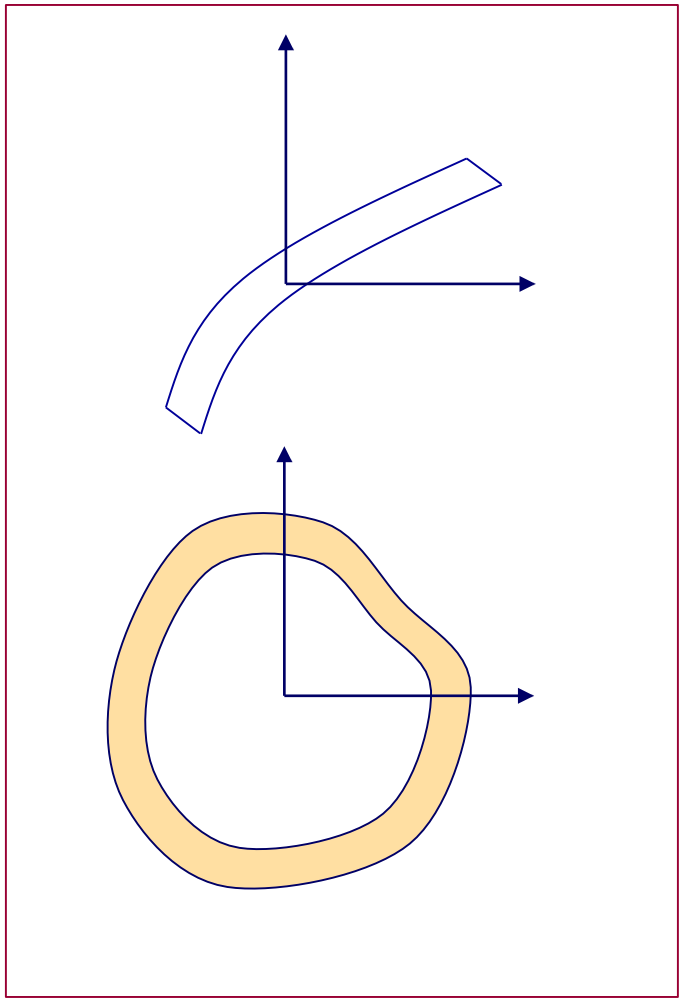
$$\sigma_z(x,y,z) = \frac{N}{A} + \frac{y \cdot I_y - x \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \cdot M_x + \frac{y \cdot P_{xy} - x \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \cdot M_y$$

## Shear stresses

$$q(s) = q(0) - K_y \cdot Q_y + -K_x \cdot Q_x$$

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# Summary



**Open section**

$$q(s) = q(0) - K_y \cdot Q_y + -K_x \cdot Q_x$$

**Closed sections**

$$q(s) = q(s)_{open} + q(0)$$

$$q(0) = -\frac{\int_0^S \frac{q(s)_{open}}{G \cdot e(s)} \cdot ds}{\int_0^S \frac{1}{G \cdot e(s)} \cdot ds}$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# References


1. **Cervera Ruiz, M. y Blanco Diaz, E. "Mecánica de Estructuras: Libro 2. Métodos de análisis"**
   Ed. UPC, 2001
   Cap.6 Esfuerzo cortante

2. **Garrido, J.A. Y Foces, A. "Resistencia de Materiales".**
   Secretariado de Publicaciones. Universidad de Valladolid. 1994
   Cap.4 Tracción (compresión)-flexión en barras esbeltas de sección de pared delgada

3. **Ortiz Berrocal, L. "Resistencia de Materiales".**
   Ed. McGraw Hill. 2002
   Cap.4 Teoría general de la flexión. Análisis de tensiones


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# References

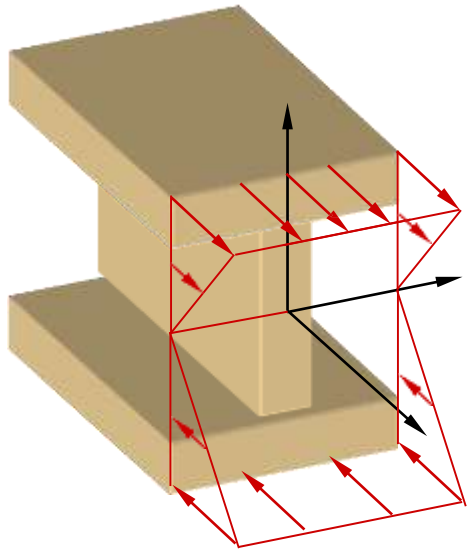

1. **Gere, J.M. "Resistencia de Materiales"**
   **Thomson. 2002**
   Cap.1 Tracción, compresión y cortante
   Cap.4 Esfuerzos cortantes y momentos flexionantes
   Cap.5 Tensiones en vigas (temas básicos)
   Cap.6 Tensiones en vigas (temas avanzados)

2. **Benham, P.P. Y Crawford. "Mechanics of engineering materials"**
   **Logman Scientific & Technical, 1987**
   Cap.6 Bending: Stress


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# References

**Mechanical Design Group**
**University of Massachusetts Amherst**
**http://www.ecs.umass.edu/mie/labs/mda/dlib/peter/statload.htm**



**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Bachelor in Aerospace Engineering



Universidad Carlos III de Madrid

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## Department of Continuum Mechanics and Structural Engineering

# Aerospace Structures

## Chapter 2. Bending, shear and torsion of thin-walled beams

### Bending and shear of open and closed, thin-walled beams (II)

# CHAPTER 2. Bending, shear and torsion of thin-walled beams

## Bending and shear of open and closed, thin-walled beams (II)

**OBJETIVES**

- Define and determine the shear centre
- Estimate the shear stress distribution by a alternate method to the proposed in section 2
- Calculate the shear stress distribution on multiple cell sections

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

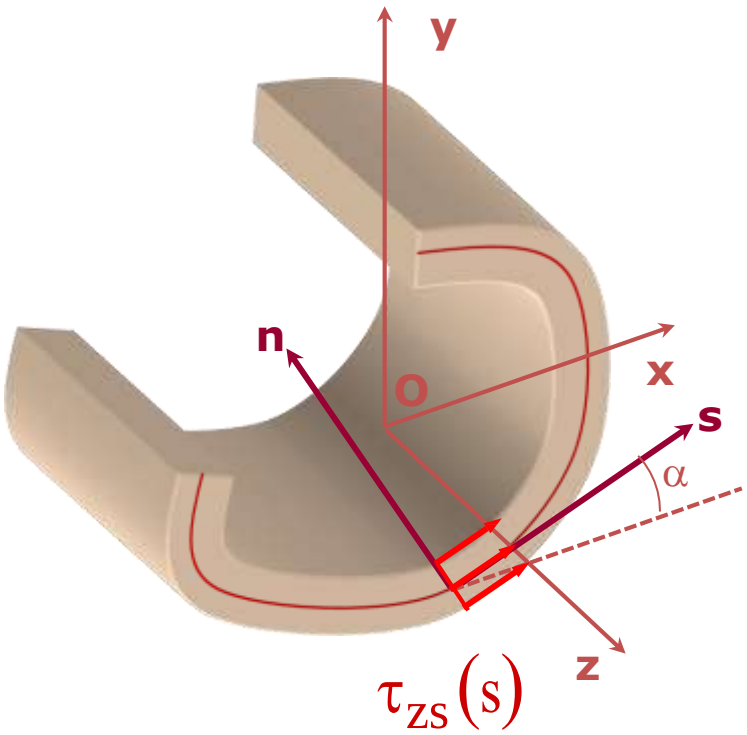
# CHAPTER 2. Bending, shear and torsion of thin-walled beams

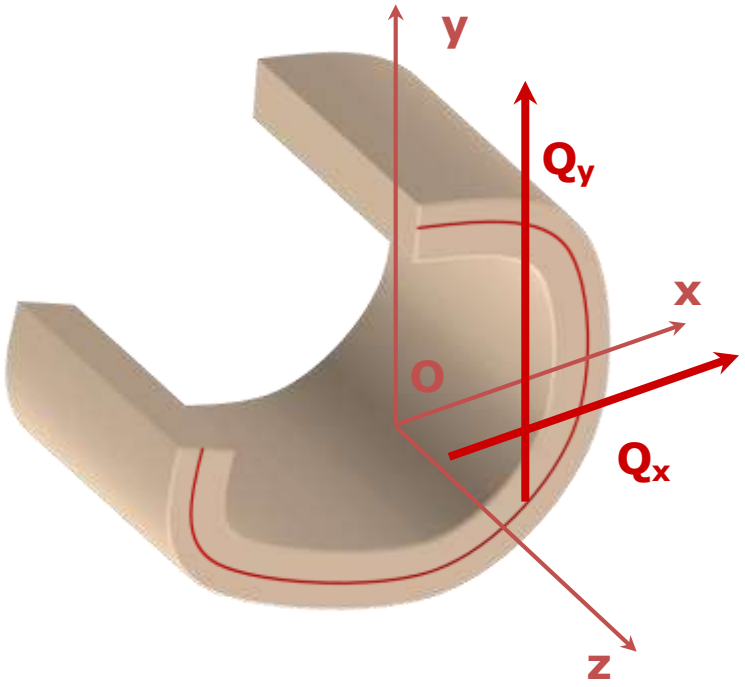
## Bending and shear of open and closed, thin-walled beams (II)

1. Concept of shear centre
2. Alternate method to calculate shear stress
3. Stress field on multiple cell sections
4. Combined open and closed section
5. Summary
6. References

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# Concept of Shear centre



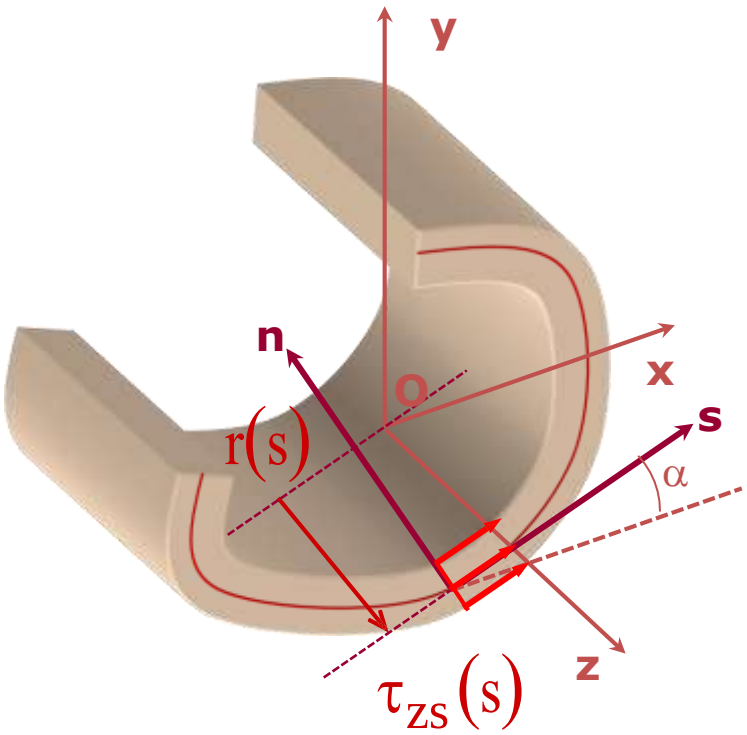



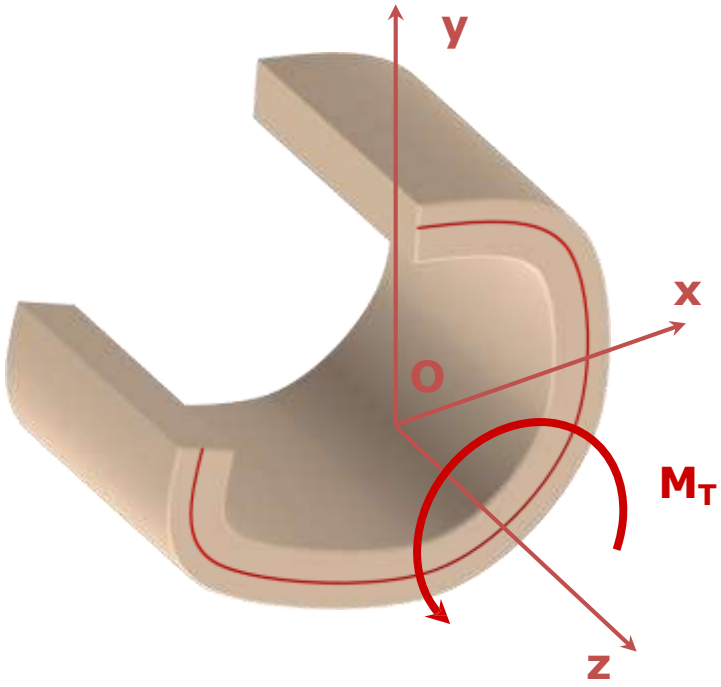

$$\int_0^{S_p} \tau_{zs}(s)\cdot\cos\alpha\cdot e(s)\cdot ds = Q_x$$

$$\int_0^{S_p} \tau_{zs}(s)\cdot\mathrm{sen}\alpha\cdot e(s)\cdot ds = Q_y$$

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# Concept of Shear centre



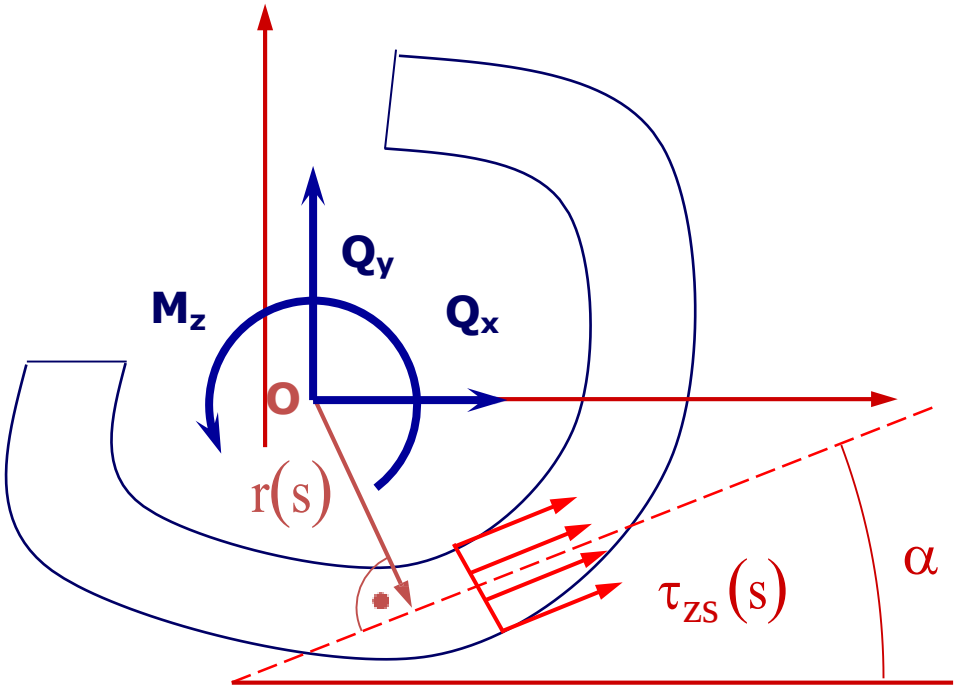




$$\int_0^{S_p} \tau_{zs}(s)\cdot e(s)\cdot r(s)\cdot ds = M_T$$

A torsion moment is induced by transverse loads

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# Concept of Shear centre




$\tau_{zs}(s)$

Shear force, $Q_x$
Shear force, $Q_y$
Torsion moment, $M_z$

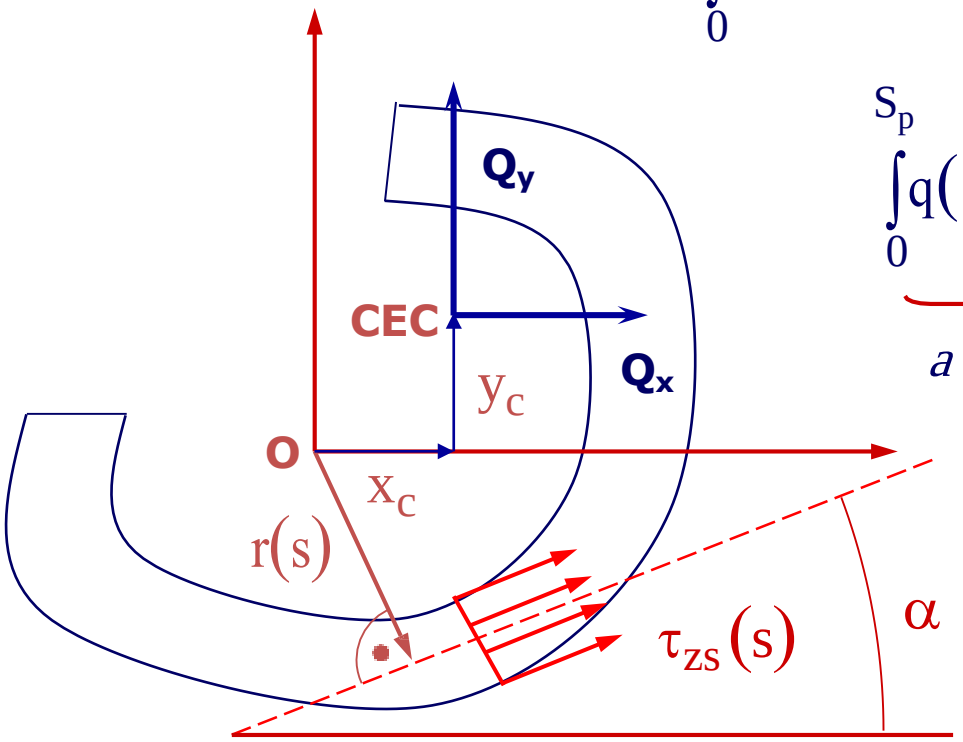
**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Concept of Shear centre

$$\int_0^{S_p} \tau_{zs}(s)\cdot e(s)\cdot r(s)\cdot ds = Q_y\cdot x_c - Q_x\cdot y_c$$

$$\int_0^{S_p} q(s)\cdot r(s)\cdot ds = Q_y\cdot x_c - Q_x\cdot y_c$$

$$\underbrace{a\cdot Q_y + b\cdot Q_x}$$




$$x_c = a$$

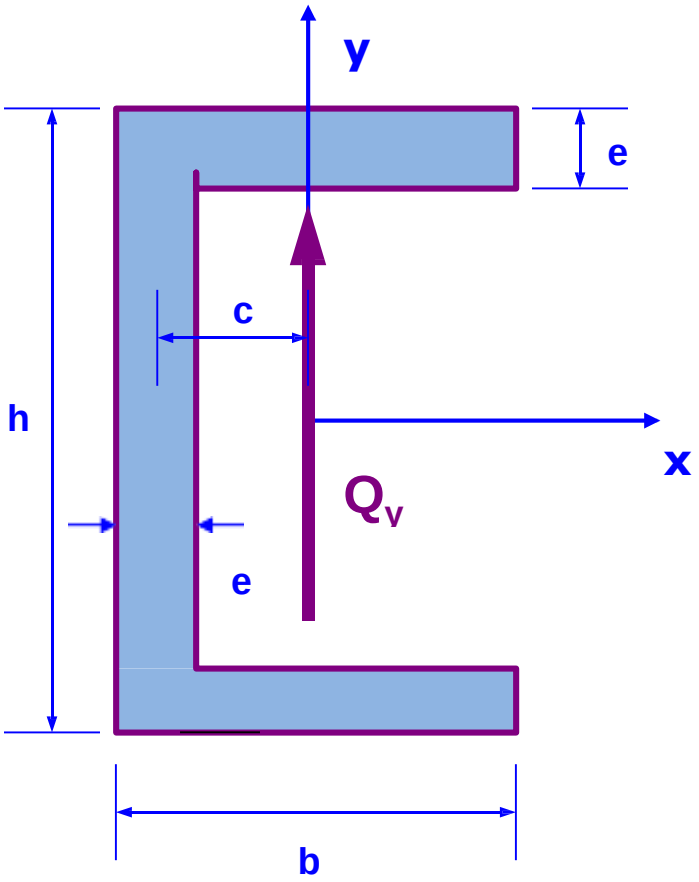
$$y_c = -b$$

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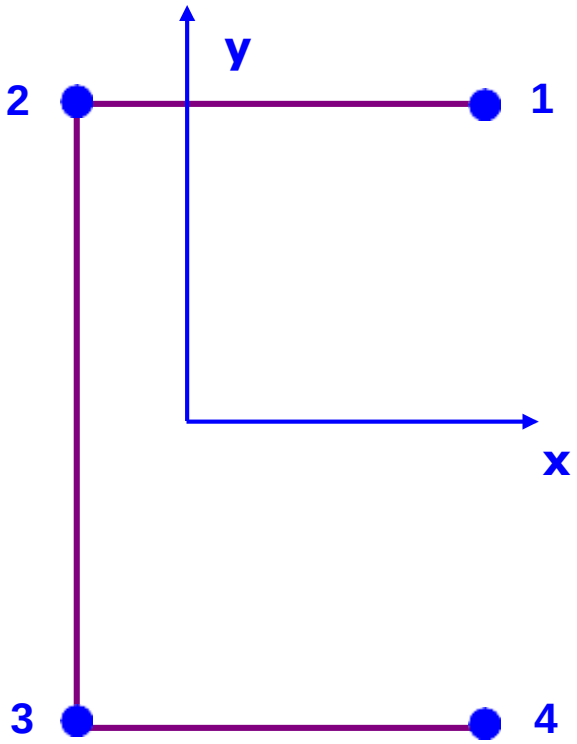
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# Concept of Shear centre

**Example**



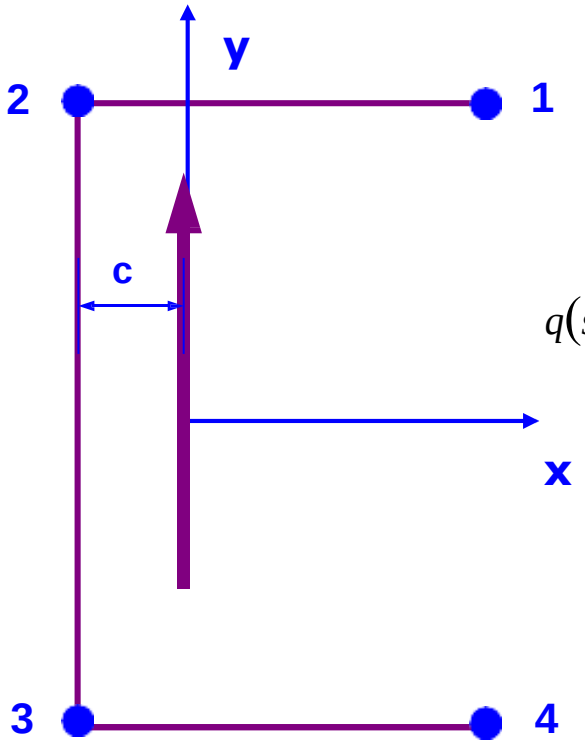

$$e = 1 \text{ cm}$$
$$b = 10 \text{ cm}$$
$$h = 20 \text{ cm}$$




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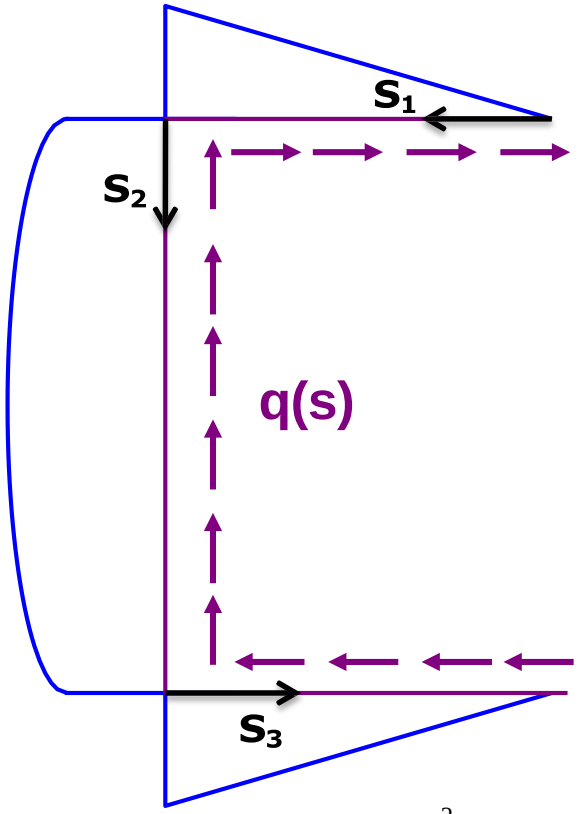
# Concept of Shear centre

## Example




$$q(s) = -\frac{Q_y \cdot e}{2 \cdot I_x}\left[h \cdot b + hs - s_2^2\right]$$

$$q(s) = -\frac{Q_y \cdot e \cdot h}{2 \cdot I_x} s_1$$

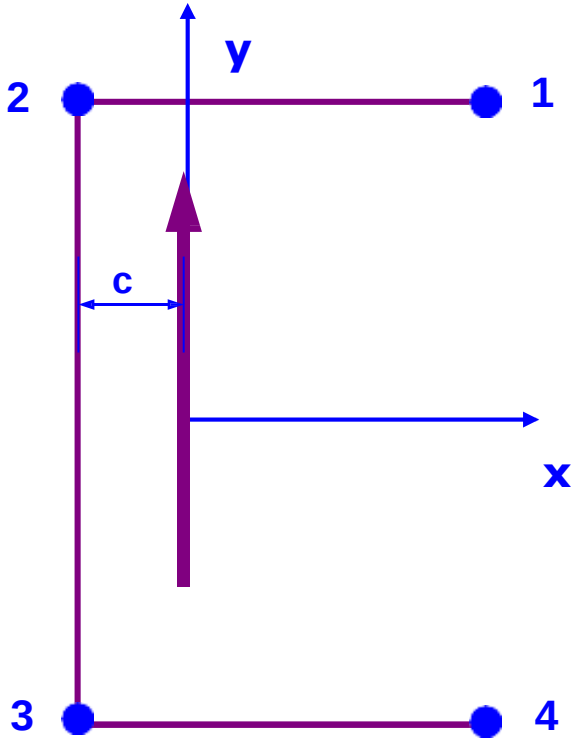



$$q(s) = -\frac{Q_y \cdot e \cdot h^2}{2 \cdot I_x}\left[b - s_3\right]$$

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# Concept of Shear centre

**Example**




$$\int_0^{S_p} q(s) \cdot r(s) \cdot ds = -Q_y \cdot c$$

$$\int_0^{S_p} q(s) \cdot r(s) \cdot ds = 2 \cdot \int_0^{b} q(s_1) \cdot \frac{h}{2} \cdot ds_1 = \frac{Q_y \cdot e \cdot h^2 \cdot b^2}{4 \cdot I_x}$$

$$\frac{Q_y \cdot e \cdot h^2 \cdot b^2}{4 \cdot I_x} = -Q_y \cdot c$$

$$c = \frac{e \cdot h^2 \cdot b^2}{4 \cdot I_x} = -3.75cm$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Alternative method to calculate Shear stresses in closed sections

The solution for a shear loaded closed section beam follows a similar pattern to that described in previous slides but with two important differences.

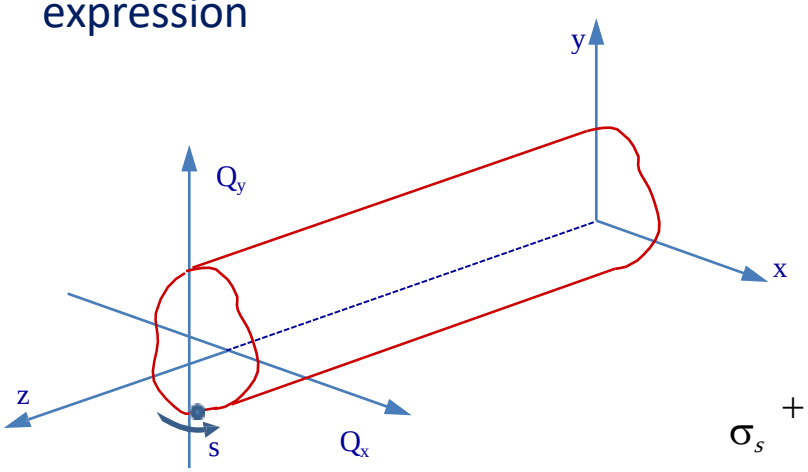
- First, the shear flow includes the effect of shear force and the induced torsional moment (if shear force is not applied at shear centre)

- Secondly, the shear loads may be applied through points in the cross-section other than the shear centre so that torsional as well as shear effects are included. This is possible since shear stresses produced by torsion in closed section beams have exactly the same form as shear stresses produced by shear, unlike shear stresses due to shear and torsion in open section beams.

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

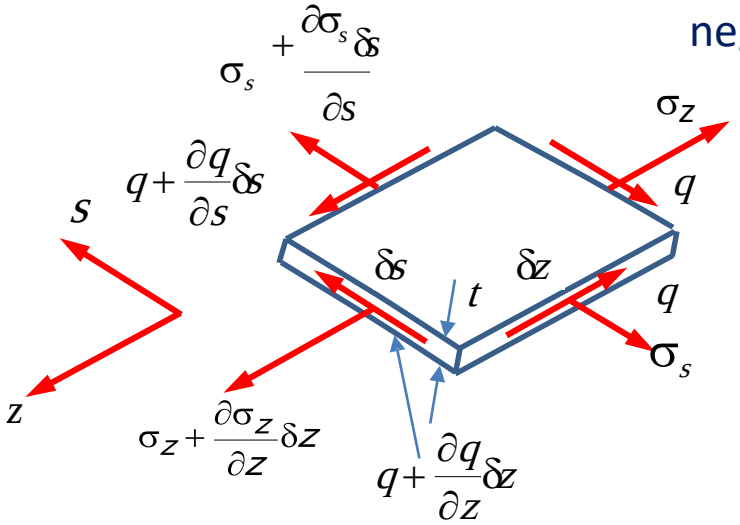
# Alternative method to calculate Shear stresses in closed sections

The shear flow can be calculated including the torsional as well as shear effects in one expression




In a beam subjected to transversal loads, both shear flow and normal stress appear

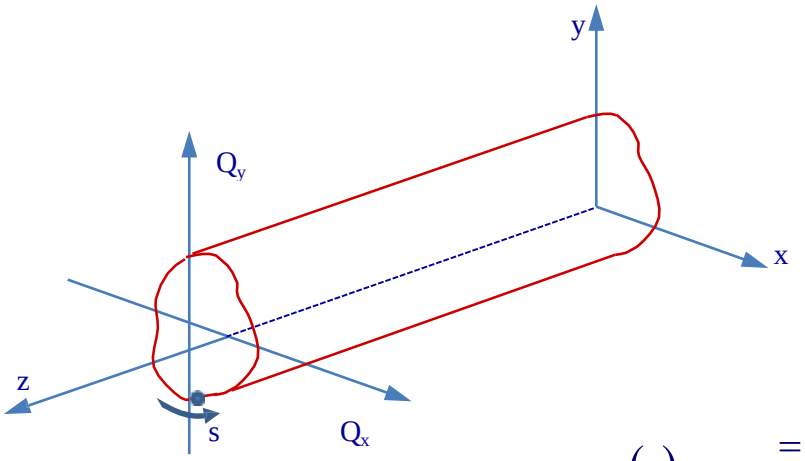
The equilibrium of a differential element in the z direction, neglecting body forces, yields to:




$$\frac{\partial q}{\partial s} + t \cdot \frac{\partial \sigma_z}{\partial s} = 0$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Alternative method to calculate Shear stresses in closed sections




$$\frac{\partial q}{\partial s} + t \cdot \frac{\partial \sigma_z}{\partial s} = 0$$

Integrating as before

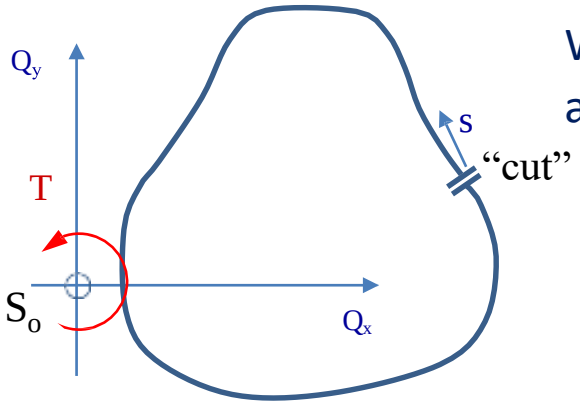
$$q(s) - q_{s_0} = -\left( \frac{Q_x \cdot I_x - Q_y \cdot P_{xy}}{I_x \cdot I_y - F_{xy}^2} \right) \cdot \int_0^S x \cdot t \cdot ds - \left( \frac{Q_y \cdot I_y - Q_x \cdot P_{xy}}{I_x \cdot I_y - F_{xy}^2} \right) \cdot \int_0^S y \cdot t \cdot ds$$

$$q(s) = -\left( \frac{Q_x \cdot I_x - Q_y \cdot P_{xy}}{I_x \cdot I_y - F_{xy}^2} \right) \cdot \int_0^S x \cdot t(s) \cdot ds - \left( \frac{Q_y \cdot I_y - Q_x \cdot P_{xy}}{I_x \cdot I_y - F_{xy}^2} \right) \cdot \int_0^S y \cdot t(s) \cdot ds + q_{s_0}$$

$$q(s) = q_b(s) + q_{s_0}$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Alternative method to calculate Shear stresses in closed sections




We obtain $q_b$ by supposing that the closed beam section is cut at some convenient point thereby producing an open section

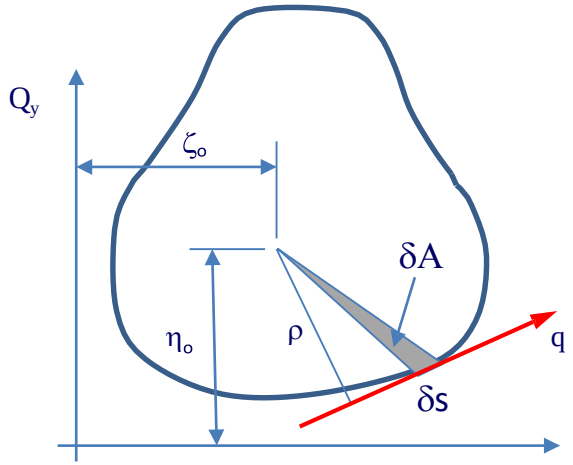
$$q_b = -\left(\frac{Q_x \cdot I_x - Q_y \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2}\right)\cdot\int_0^S x\cdot t(s)\cdot ds - \left(\frac{Q_y \cdot I_y - Q_x \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2}\right)\cdot\int_0^S y\cdot t(s)\cdot ds$$

The value of shear flow at the cut (s = 0) is then found by equating applied and internal moments taken about some convenient moment centre

$$Q_x \cdot \eta_o - Q_y \cdot \xi_o = \oint q(s)\cdot p(s)\cdot ds$$

$$Q_x \cdot \eta_o - Q_y \cdot \xi_o = \oint q_b(s)\cdot p(s)\cdot ds + q_{s_o}\cdot\oint p(s)\cdot ds$$

$$q_{s_o} = \frac{Q_x \cdot \eta_o - Q_y \cdot \xi_o - \oint q_b(s)\cdot p(s)\cdot ds}{2\cdot\Omega}$$



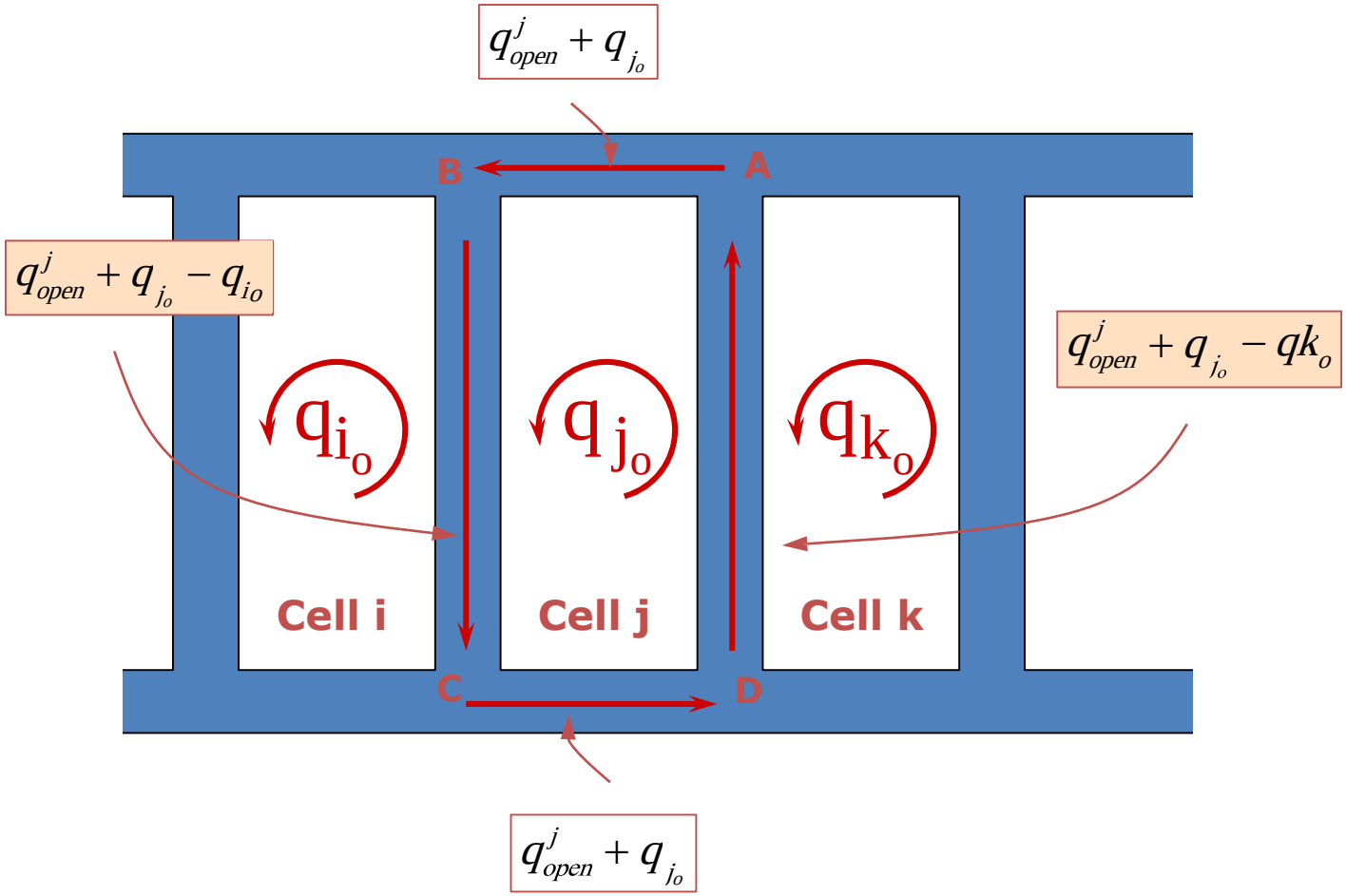

$\delta$

$$2\cdot\Omega = \oint p(s)\cdot ds$$

$\Omega$ is the area enclosed by the mid-line of the beam section wall

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Multi-cell closed sections




**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Multi-cell closed sections

Solution process for a section with n cells:

- The section in divided by n-cuts

$$q^{j}_{open}$$

- Each cell is solved as open

- Imposing in each cell:

$$\int_{0}^{S_p} \gamma_{zs} \cdot ds = 0$$

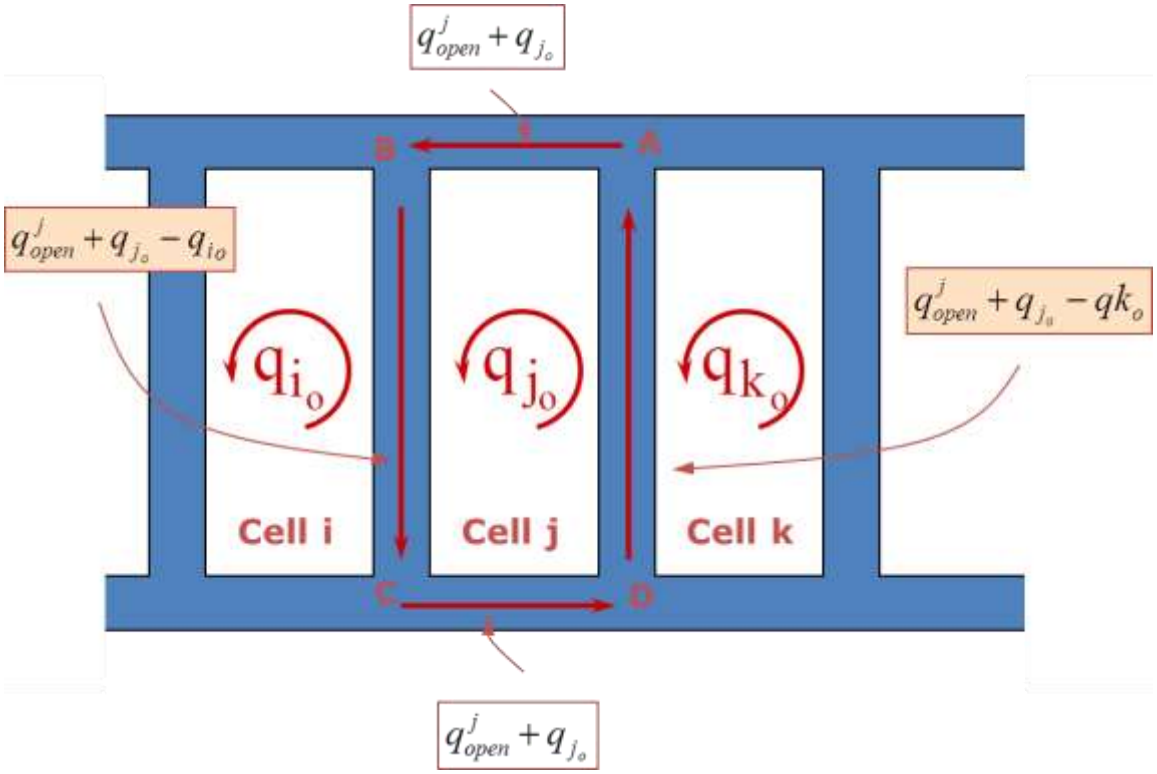
System of n-equations with n-unknowns

$$\int_{A}^{B} \frac{\left(q^{j}_{open} + q_{j_o}\right)}{G} \cdot ds + \int_{D}^{C} \frac{\left(q^{j}_{open} + q_{j_o} - q_{i^o}\right)}{G} \cdot ds +$$

$$\int_{C}^{D} \frac{\left(q^{j}_{open} + q_{j_o}\right)}{G} \cdot ds + \int_{A}^{E} \frac{\left(q^{j}_{open} + q_{j_o} - q_{k_o}\right)}{G} \cdot ds = 0$$

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# Multi-cell closed sections



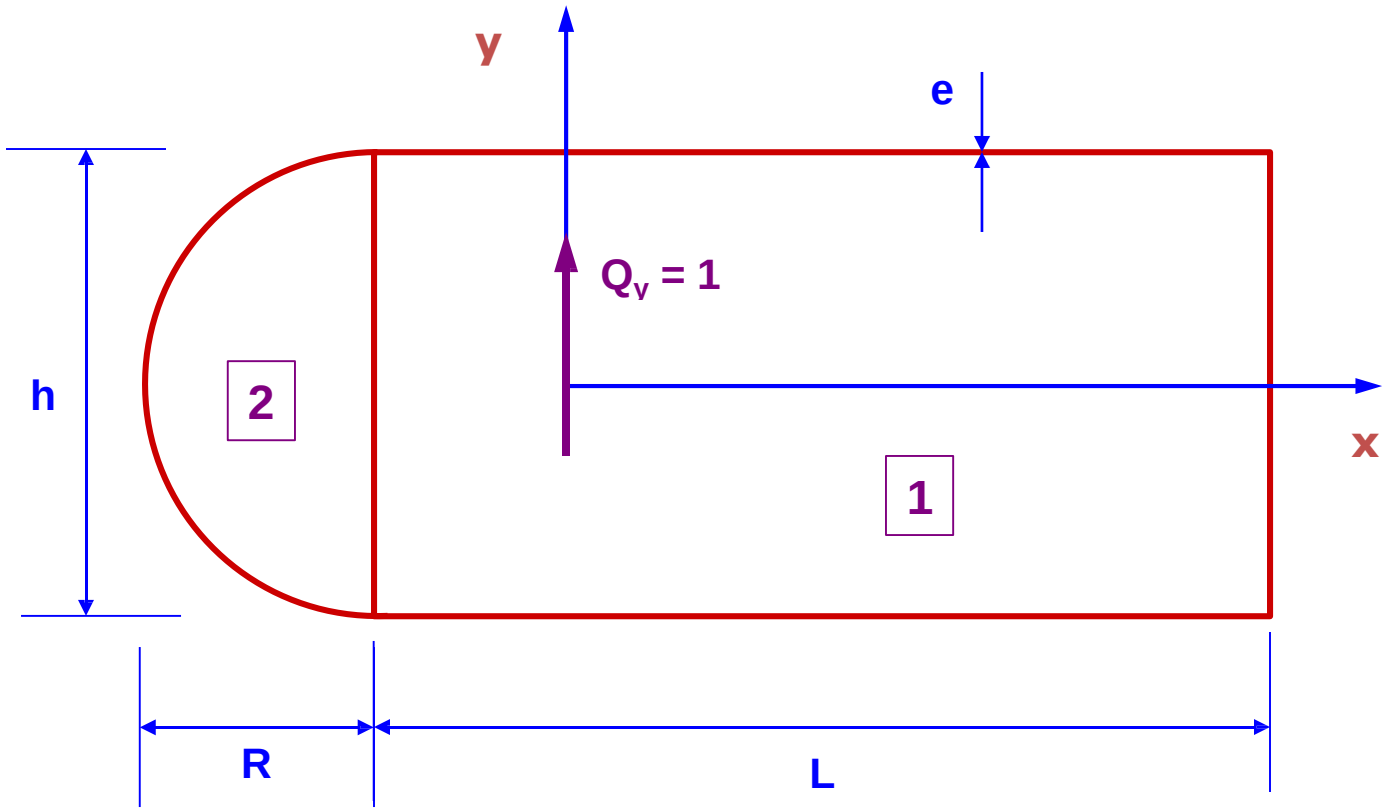

$$\int_A^B \frac{q^j_{open} + q_{j_o}}{G} \cdot ds + \int_B^C \frac{q^j_{open} + q_{j_o} - q_{i_o}}{G} \cdot ds +$$

$$\int_D^A \frac{q^j_{open} + q_{j_o}}{G} \cdot ds + \int_C^D \frac{q^j_{open} + q_{j_o} - q_{k_o}}{G} \cdot ds = 0$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Multi-cell closed sections

Example




$e = 0{,}5$ cm

$L = 50$ cm

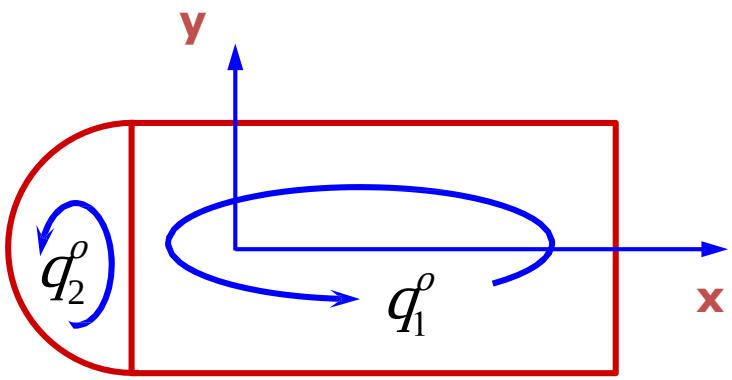
$h = 12$ cm

$R = 6$ cm

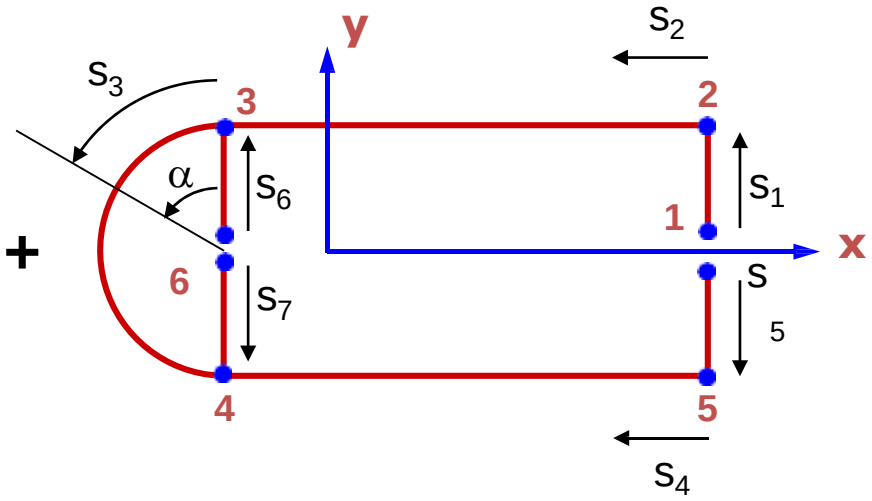
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# Multi-cell closed sections

Example




**Cuts at points 1 and 6**



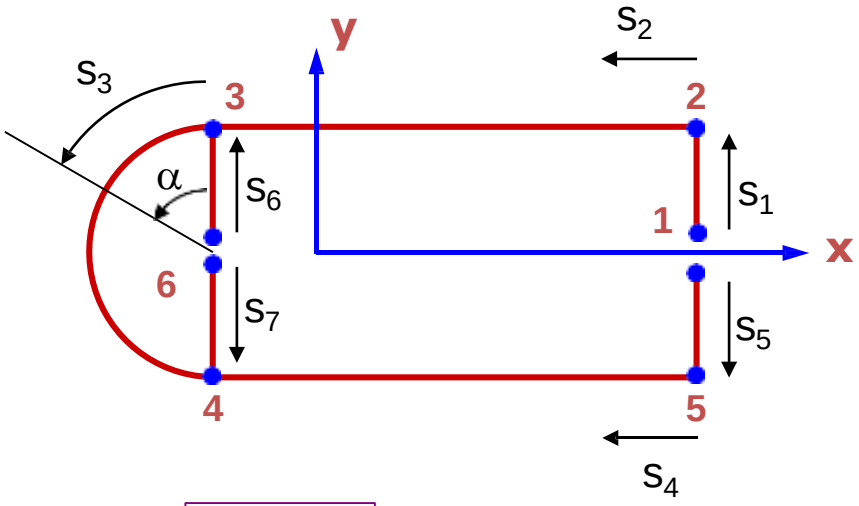

$$m_{ex}^{*} = \int_0^s y(s) \cdot e(s) \cdot ds$$

$$q_a(s) = -\frac{m_{ex}^{*}(s)}{I_x} \cdot Q_y$$

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# Multi-cell closed sections

## Example




**Skin 12**

$$m^*_{ex} = \frac{e \cdot s_1^2}{2}$$

$$q_a(s) = -\frac{e \cdot s_1^2}{2 \cdot I_x}$$

**Skin 23**

$$m^*_{ex} = \frac{e \cdot h^2}{8} + \frac{h \cdot e \cdot s^2}{2}$$

$$q_a(s) = -\frac{e \cdot h}{2 \cdot I_x} \cdot \left( s_2 + \frac{h}{4} \right)$$

**Skin 63**

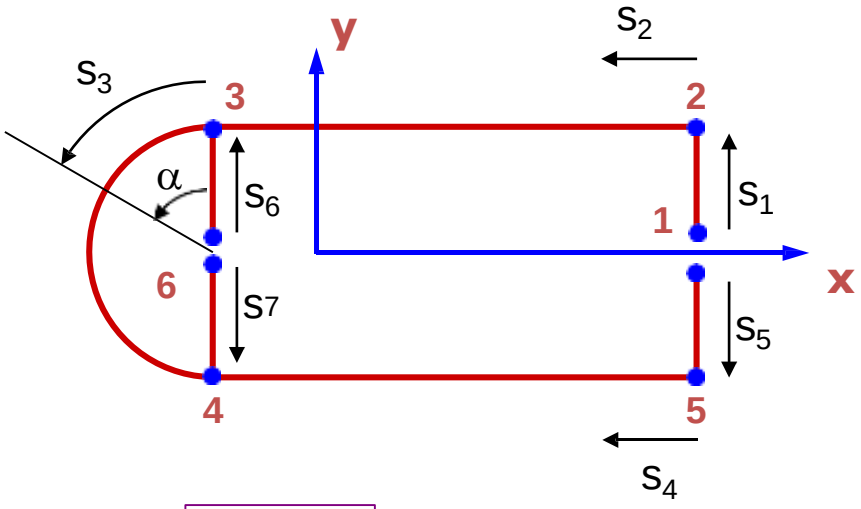
$$m^*_{ex} = \frac{e \cdot s^2}{6}$$

$$q_a(s) = -\frac{e \cdot s_6^2}{2 \cdot I_x}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Multi-cell closed sections

## Example




**Skin 15**

$$m_{ex}^{*} = -\frac{e \cdot s^{2}}{2}$$

$$q_a(s) = \frac{e \cdot s^{2}}{2 \cdot I_x}$$

**Skin 54**

$$m_{ex}^{*} = -\left(\frac{e \cdot h^{2}}{8} + \frac{h \cdot e \cdot s_4}{2}\right)$$

$$q_a(s) = \frac{e \cdot h}{2 \cdot I_x} \cdot \left(s_4 + \frac{h}{4}\right)$$

**Skin 64**

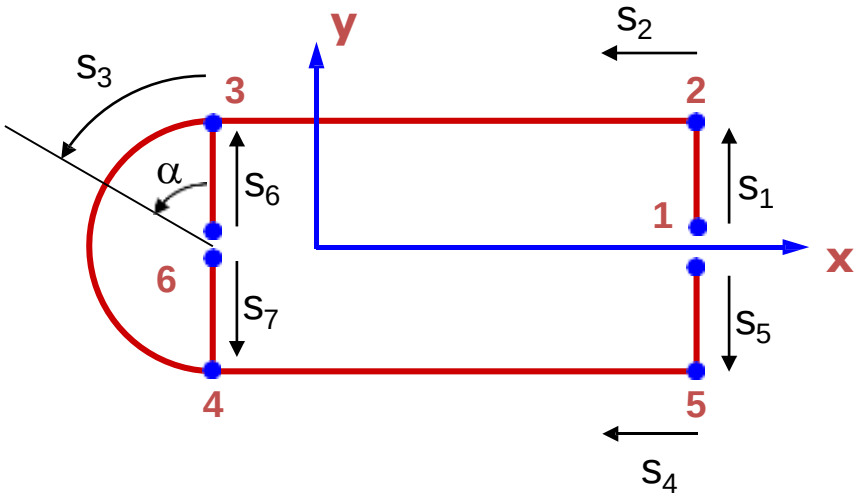
$$m_{ex}^{*} = -\frac{e \cdot s_7^{2}}{2}$$

$$q^{a}(s) = \frac{e \cdot s_7^{2}}{2 \cdot I_x}$$

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# Multi-cell closed sections

## Example




**Skin 34**

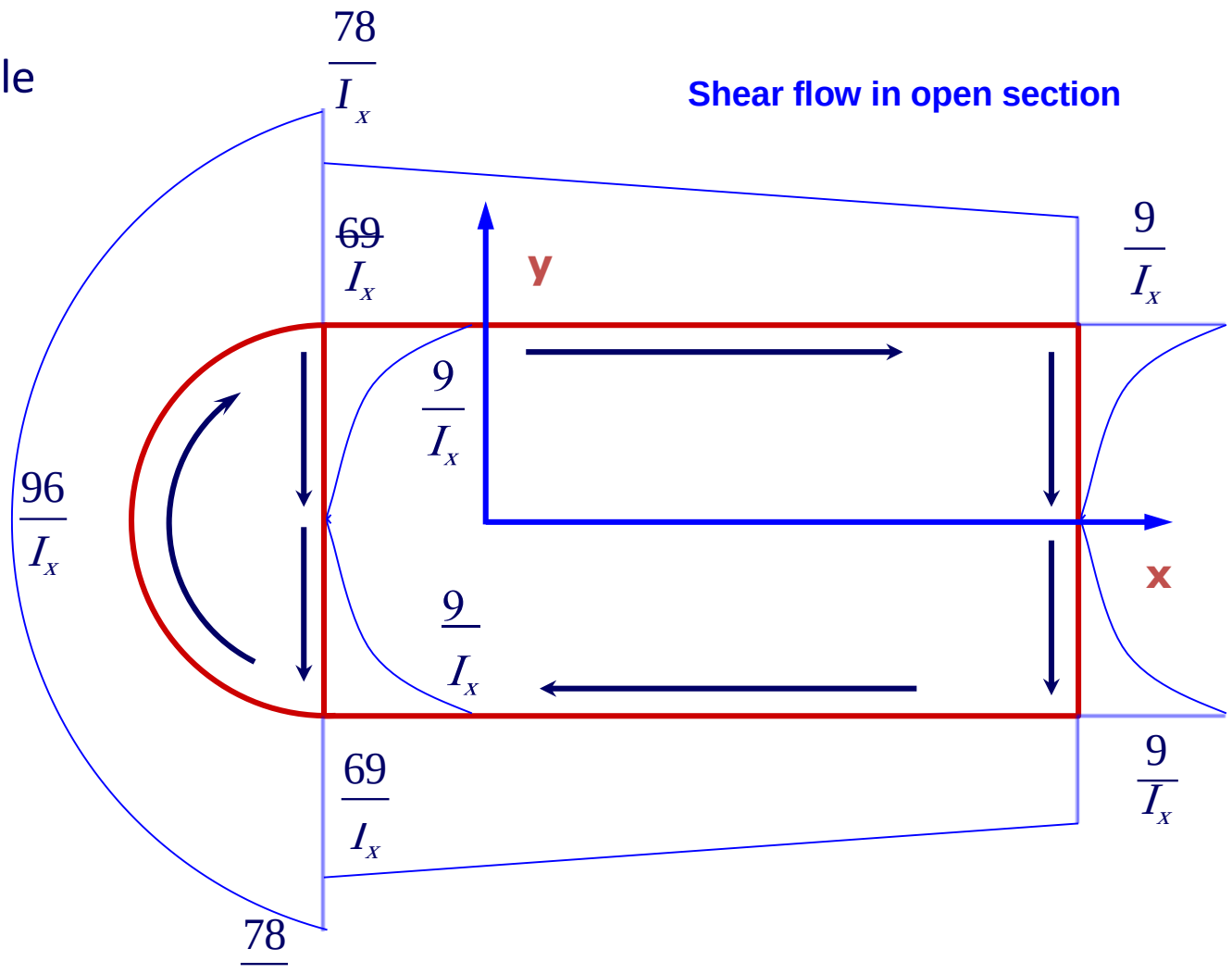
$$m^*_{ex} = \frac{e \cdot h^2}{4} + \frac{h \cdot e \cdot L}{2} + \int_0^{\alpha} R \cdot \cos\alpha \cdot e \cdot R \cdot d\alpha = \frac{e \cdot h^2}{4} + \frac{h \cdot e \cdot L}{2} + R^2 \cdot e \cdot sen\alpha$$

$$q_a(\alpha) = -\frac{1}{I_x} \cdot \left| \left( \frac{e \cdot h^2}{4} + \frac{h \cdot e \cdot L}{2} + R^2 \cdot e \cdot sen\alpha \right) \right|$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Multi-cell closed sections

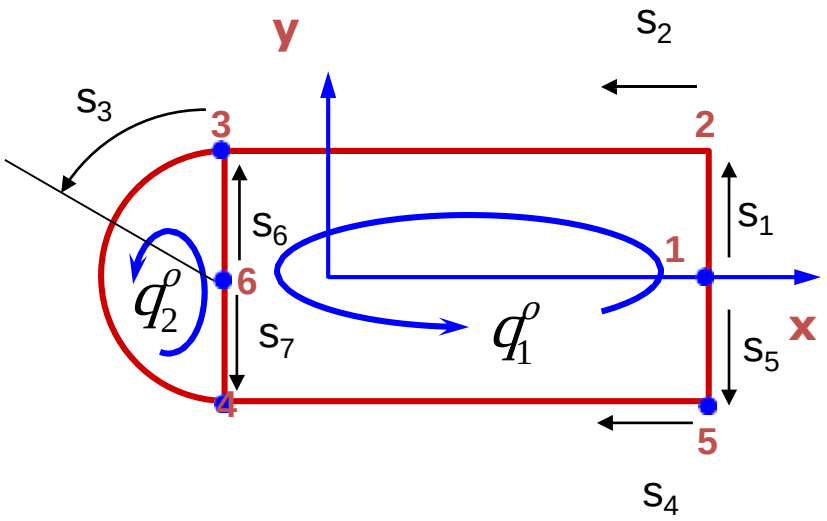
Example




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# Multi-cell closed sections

## Example




**Skin 12** $q(s) = q_a^{12}(s_1) + q_1^o$

**Skin 23** $q(s) = q_a^{23}(s_2) + q_1^o$

**Skin 63** $q(s) = q_a^{63}(s_6) + q_2^o - q_1^o$

**Skin 64** $q(s) = q_a^{64}(s_7) + q_1^o - q_2^o$

**Skin 15** $q(s) = q_a^{15}(s_5) - q_1^o$

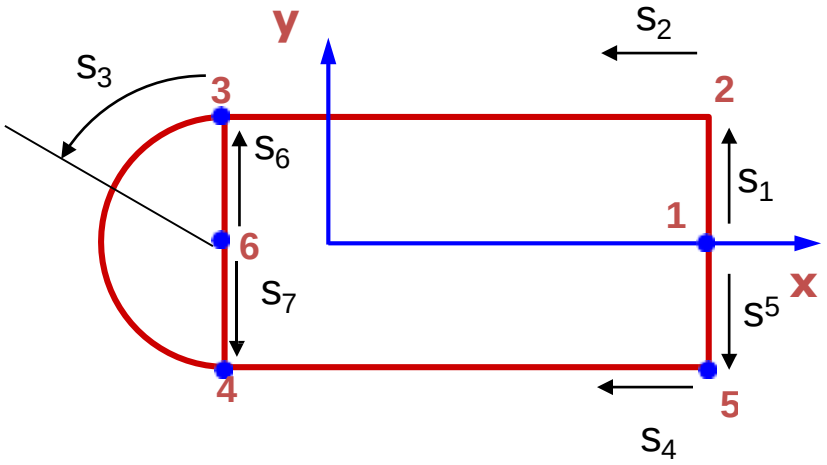
**Skin 34** $q(s) = q_a^{34}(\alpha) + q_2^o$

**Skin 54** $q(s) = q_a^{54}(s_4) - q_1^o$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Multi-cell closed sections

## Example




**Imposing in each cell:**

$$\int_0^{S_p} \frac{q(s)}{e(s)} \cdot ds = 0$$

**In cell 1**

$$\int_0^{h/2} \left( q_a^{12}(s_1) + q^0 \right)_1 \cdot ds_1 + \int_0^{L} \left( q_a^{23}(s_2) + q^0 \right)_2 \cdot ds_2 +$$

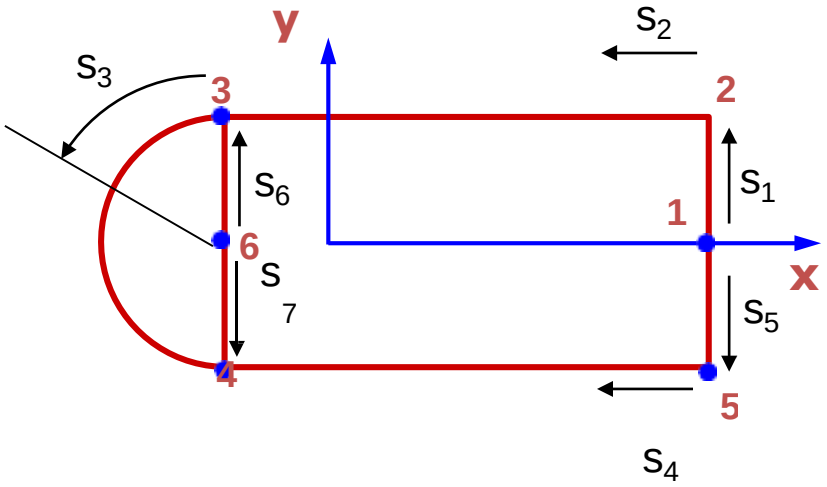
$$- \int_0^{h/2} \left( q_a^{63}(s_6) + q^0 - q^0 \right)_6 \cdot ds_6 + \int_0^{h/2} \left( q_a^{64}(s_7) - q^0 + q^0 \right)_7 \cdot ds_7 +$$

$$- \int_0^{L} \left( q_a^{45}(s_4) - q^0 \right)_4 \cdot ds_4 - \int_0^{L} \left( q_a^{51}(s_5) - q^0 \right)_5 \cdot ds_5$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Multi-cell closed sections

## Example




**In cell 2**

$$\int_0^{\pi} \left( q_a^{34}(\alpha) + q_2^0 \right) \cdot R \cdot d\alpha - \int_0^{h} \left( q_a^{64}(s_7) + q_1^0 - q_2^0 \right) \cdot ds_2 + \int_0^{h} \left( q_a^{63}(s_6) - q_1^0 + q_2^0 \right) \cdot ds_6 = 0$$

**Imposing in each cell:**

$$\oint_0^{S} \frac{q(s)}{e(s)} \cdot ds = 0$$

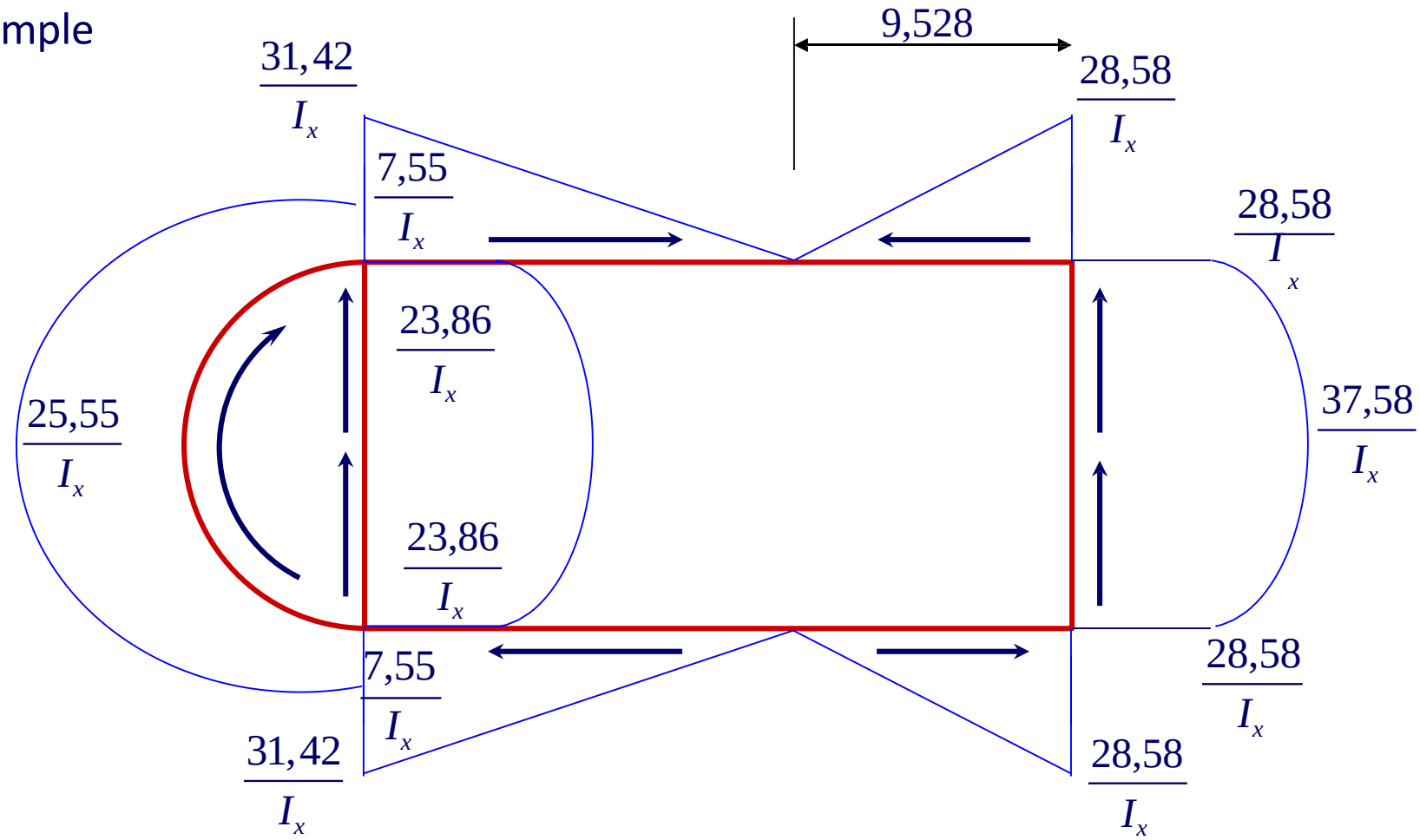
$$q_1^o = \frac{37{,}58}{I_x}$$

$$q_2^o = \frac{77{,}44}{I_x}$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Multi-cell closed sections

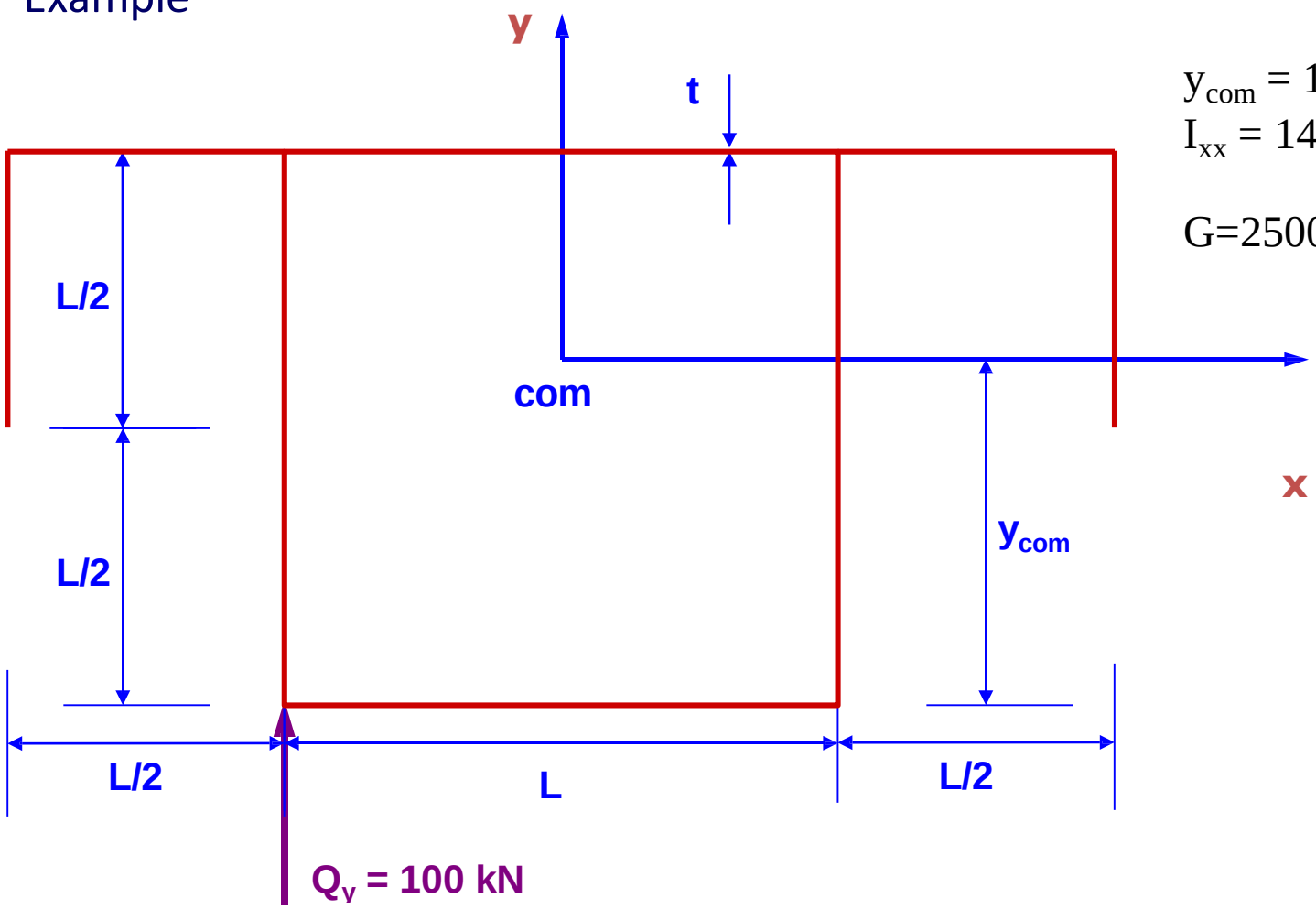
Example




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# Combined open and closed sections

Example




t = 2 mm
L = 200 mm

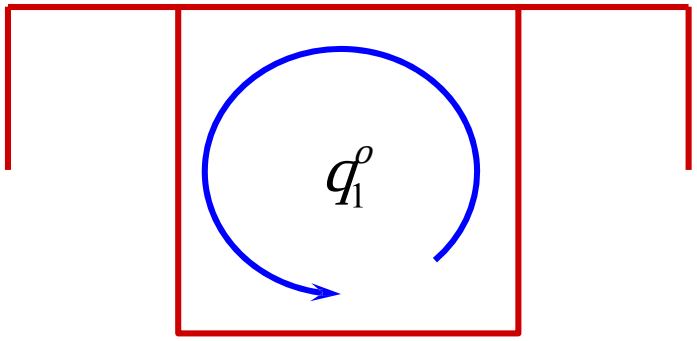
$y_{com}$ = 125 mm
$I_{xx} = 14.5 \cdot 10^6$ mm$^4$

G=25000 N/mm$^2$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

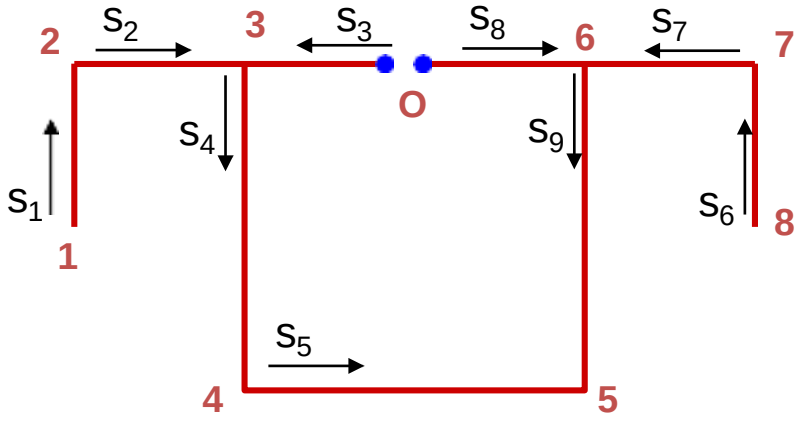
# Combined open and closed sections

Shear force $Q_y$




+

Cuts at point O



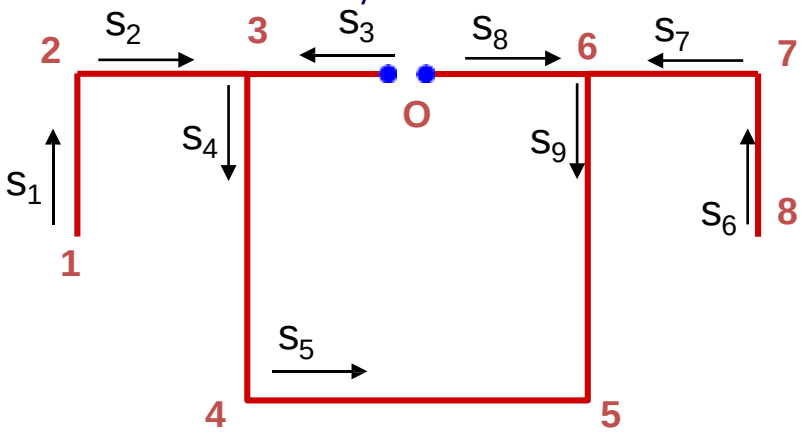

$$m_{ex}^{*} = \int_0^s y(s) \cdot e(s) \cdot ds$$

$$q_a(s) = -\frac{m_{ex}^{*}(s)}{I_x} \cdot Q_y$$

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# Combined open and closed sections

## Shear force $Q_y$




## Skin 23

$$q_{23}(s_2) = -\frac{Q_y}{I_{xx}} \int_0^{s_2} 2 \cdot 75 \cdot ds_2 - 34.5 = -34.5 - 1.04 \cdot s_2$$

$$q_{23}\left(\frac{L}{2}\right) = q_{3,23} = -138.5 N/mm$$

$$q_{67}(s_7) = q_{23}(s_2)$$

$$q_{6,67} = q_{3,23}$$

## Skin 12

$$q_{12}(s_1) = -\frac{Q_y}{I_{xx}} \int_0^{s_1} 2(-25 + s_1)ds_1 = -69 \cdot 10^{-4}\left(s_1^2 - 50 s_1\right)$$

$$q_{12}\left(\frac{L}{2}\right) = q_2 = -34.5 N/mm$$

$$q_{78}(s_6) = q_{12}(s_1)$$

$$q_7 = q_2$$

## Skin O3

$$q_{O3}(s_3) = -\frac{Q_y}{I_{xx}} \int_0^{s_3} 2 \cdot 75 \cdot ds_3 = -1.04 \cdot s_3$$

$$q_{O3}\left(\frac{L}{2}\right) = q_{3,O3} = -104 N/mm$$

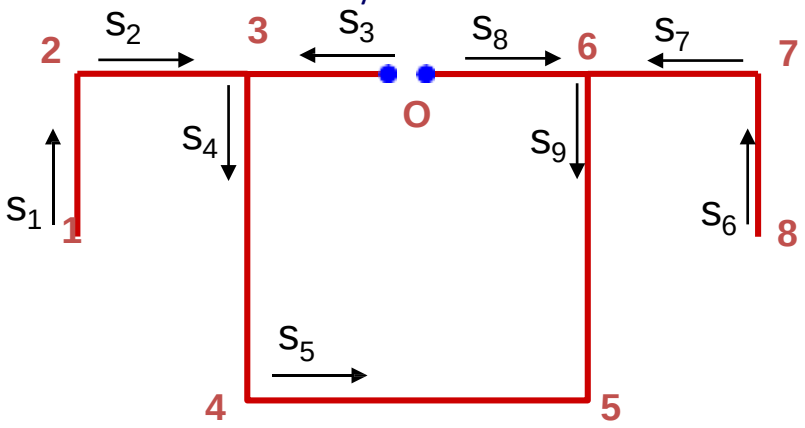
$$q_{O6}(s_8) = q_{O3}(s_3)$$

$$q_{6,O6} = q_{3,O3}$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Combined open and closed sections

Shear force $Q_y$




**Skin 45**

$$q_{45}(s_5) = -\frac{Q_y}{I_{xx}} \int_0^{s_5} 2\cdot(-125)\cdot ds_5 - 172.5 = -172.5 - 1.725\cdot s_5$$

$$q_{45}\left(\frac{L}{2}\right) = 0$$

$$q_{45}(L) = q_5 = 172.5 N/mm$$

**Skin 34**

$$q_{34}(0) = q_{3,34} = q_{3,23} + q_{3,O3} = -242.5 N/mm$$

$$q_{34}(s_4) = -\frac{Q_y}{I_{xx}} \int_0^{s_4} 2(75 - s_4) ds_4 - 242.5 = -242.5 - 1.04 s_4 + 69\cdot 10^{-4} s_4^2$$

$$q_{34}(L) = q_4 = -172.5 N/mm$$

$$q_{34,\max} = q_{34}(75mm) = -281.7 N/mm$$

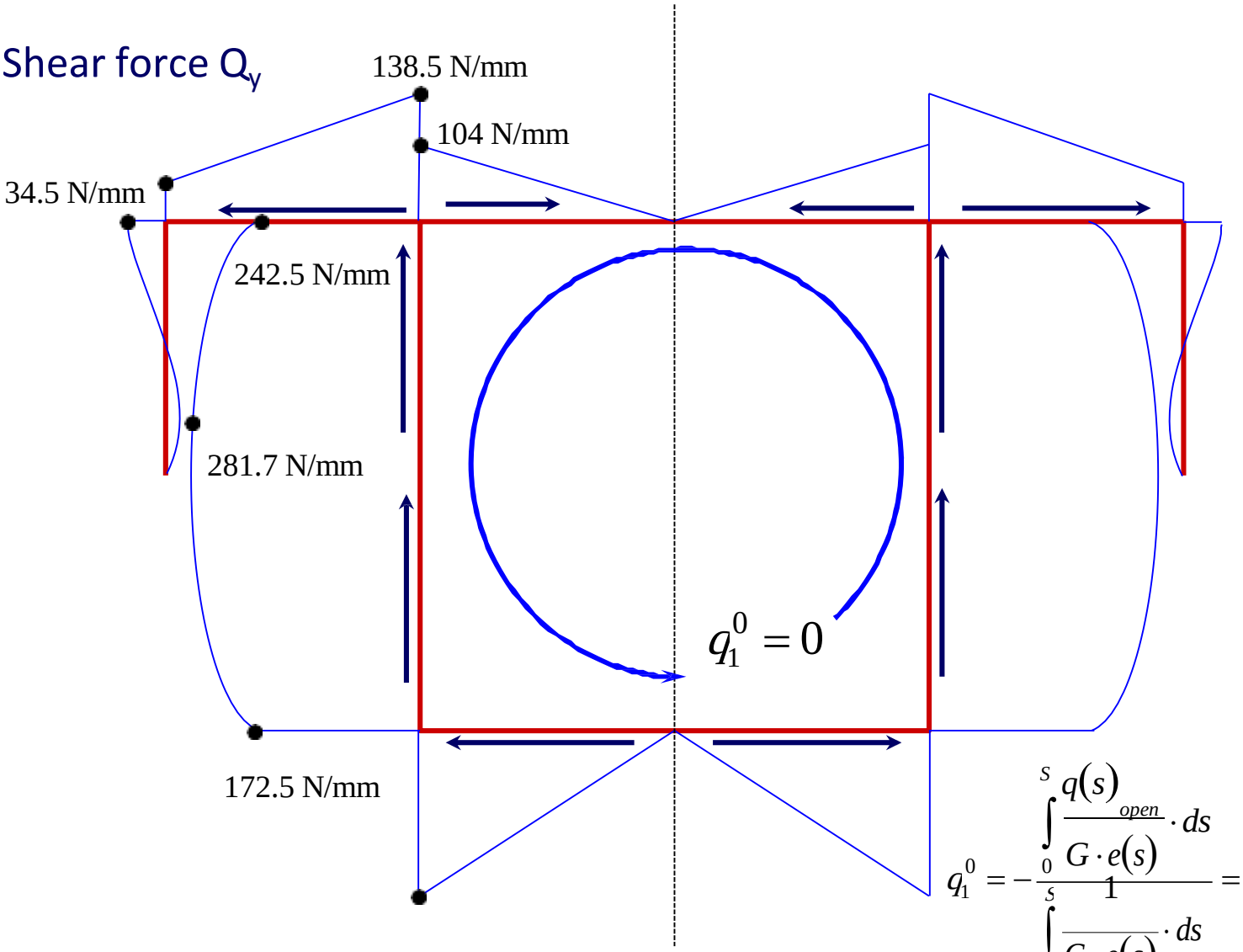
$$q_{56}(s_9) = q_{34}(s_4)$$

$$q_5 = q_4$$

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# Combined open and closed sections

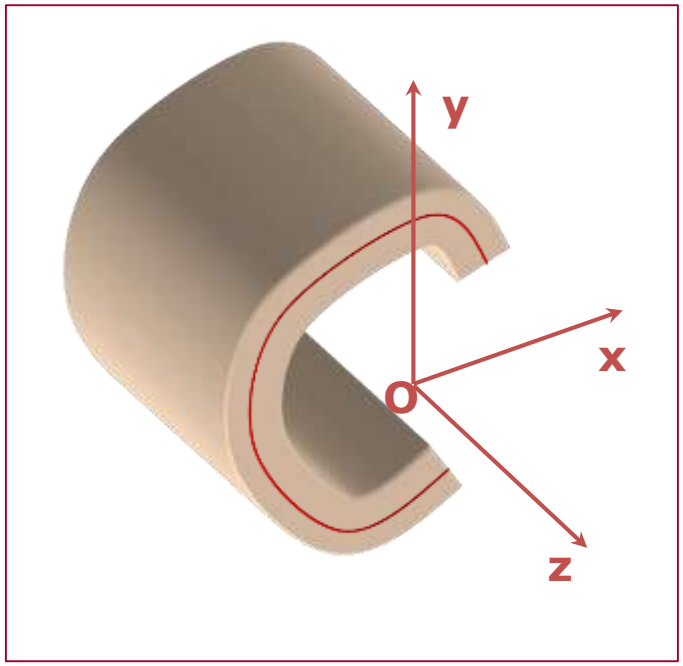
## Shear force $Q_y$




$$q_1^0 = -\frac{\int_0^S \frac{q(s)_{open}}{G \cdot e(s)} \cdot ds}{\int_0^S \frac{1}{G \cdot e(s)} \cdot ds} = -\frac{\int_0^S q(s)_{open} \cdot ds}{\int_0^S ds} = 0$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Summary




## Shear stresses

$$q(s) = q(0) - K_y \cdot Q_y + -K_x \cdot Q_x$$

$$K_y = \frac{m^*_{ex} \cdot I_y - m^*_{ey} \cdot P_{xy}}{I_x \cdot I_y - P^2_{xy}}$$

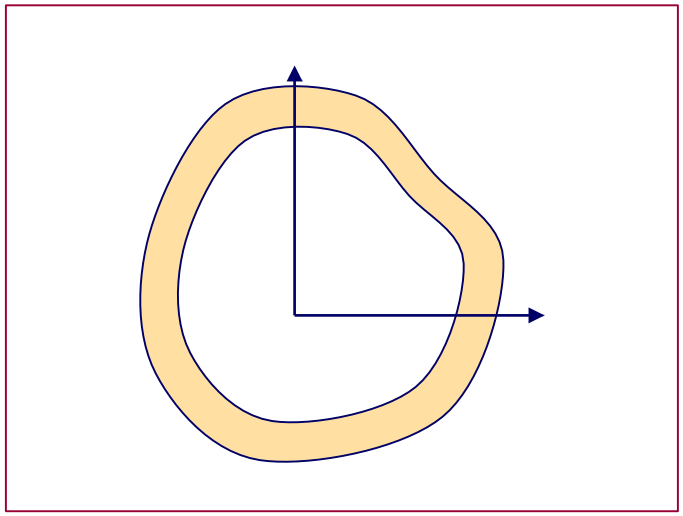
$$K_x = \frac{m^*_{ex} \cdot P_{xy} - m^*_{ey} \cdot I_x}{I_x \cdot I_y - P^2_{xy}}$$

$$m^*_{ex} = \int_{A^*} y(s) \cdot dA = \int_0^s y(s) \cdot e(s) \cdot ds$$

$$m^*_{ey} = \int_{A^*} x(s) \cdot dA = \int_0^s x(s) \cdot e(s) \cdot ds$$

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# Summary



**Single cell closed section**

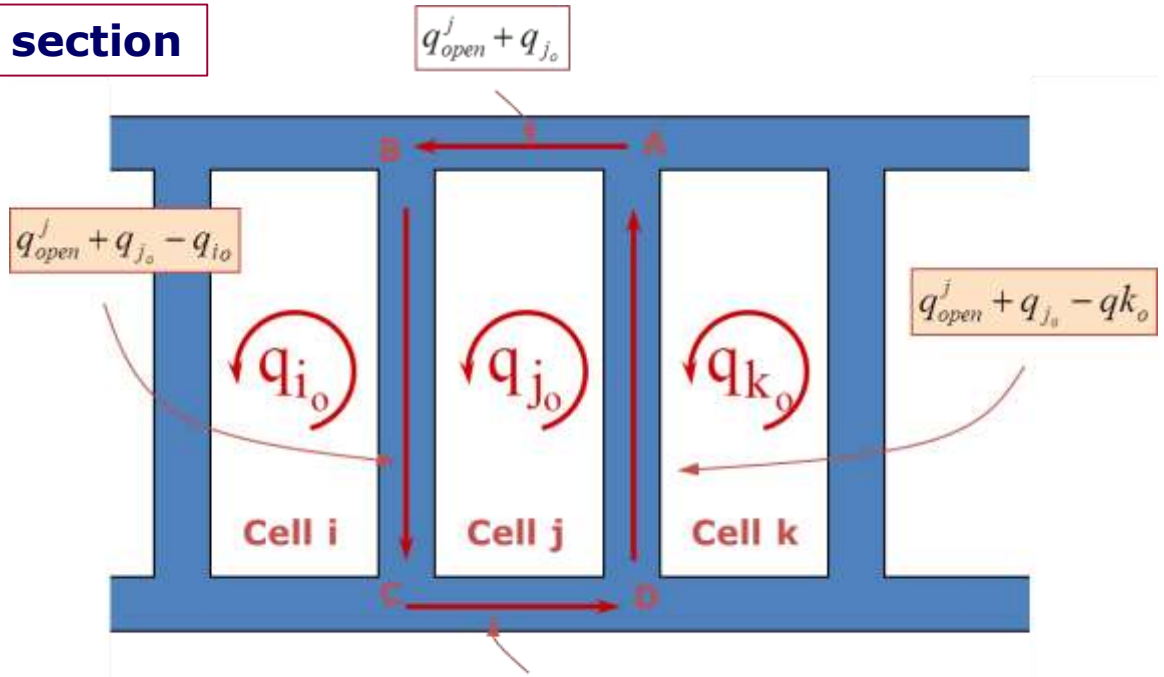
$$q(s) = q(s)_{open} + q(0)$$

$$q(0) = -\frac{\int_0^S \frac{q(s)_{open}}{G \cdot e(s)} \cdot ds}{\int_0^S \frac{1}{G \cdot e(s)} \cdot ds}$$

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# Summary

## Multi-cell section




$$\int_{A}^{B} \frac{q^{j}_{open} + q_{j_o}}{G} \cdot ds + \int_{B}^{C} \frac{q^{j}_{open} + q_{j_o} - q_{i_o}}{G} \cdot ds +$$

$$\int_{C}^{D} \frac{q^{j}_{open} + q_{j_o}}{G} \cdot ds + \int_{D}^{A} \frac{q^{j}_{open} + q_{j_o} - q_{k_o}}{G} \cdot ds = 0$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# References


1. **Megson, T.H.G. "Aircraft Structures"**
   **Elsevier. 2007**
   Cap.16 Bending of open and closed thin-walled beams
   Cap.17 Shear of beams

2. **Benham, P.P. Y Crawford. "Mechanics of engineering materials"**
   **Logman Scientific & Technical, 1987**
   Cap.5 Torsion


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# References


1. **Cervera Ruiz, M. y Blanco Diaz, E. "Resistencia de Materiales"**
   Ed. UPC, 2001
   Cap.7 Momento torsor

2. **Garrido García, J.A. Y Foces Mediavilla, A. "Resistencia de Materiales"**
   Universidad de Valladolid, 1994
   Cap.6 Torsión uniforme en barras de sección de pared delgada

3. **Ortiz Berrocal, L "Elasticidad"**
   Ed. McGraw Hill, 1998
   Cap.7 Torsión

4. **Samartin Quiroga, A. "Elasticidad"**
   Ed. Bellisco. 1990
   Cap.7 Estudio de la torsión


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**Department of Continuum Mechanics and Structural Engineering**

# Aerospace Structures

## Chapter 4. Plates and Shells

### Bending of thin plates

# Chapter 4. Plates and Shells

## Bending of thin plates

1. Introduction
2. Equilibrium equations
3. Bending equations
4. Boundary conditions
5. Example
6. References

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Chapter 4. Plates and Shells

## Bending of thin plates

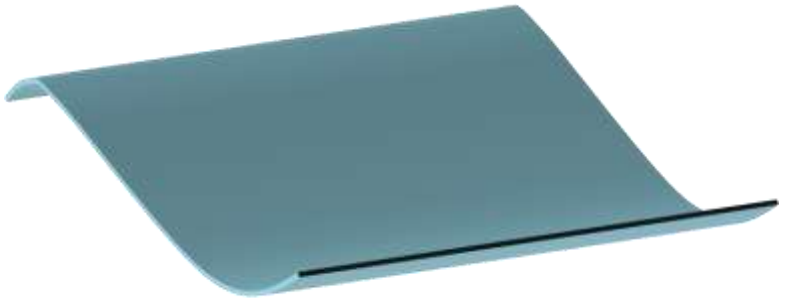
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# Introduction

## Definition

- Plate: 3D solid than be considered a 2D element because one dimension is smaller than the other two.

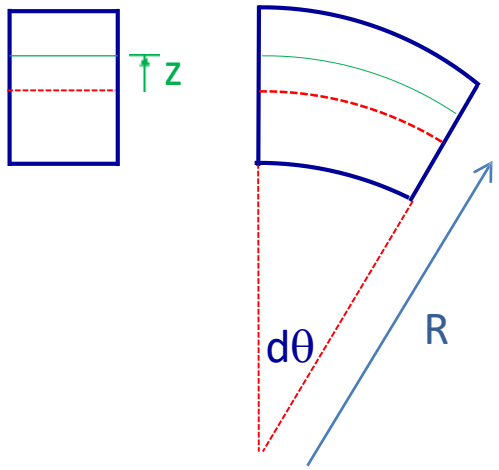


- Scientific literature use to classify plates as a function of the third dimension:
  - ✓ Thin plates (Hyp. Kirchhoff)
  - ✓ Thick plates (Hyp. Reissner Mindlin)

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# Introduction

- Thin plates (Hyp. Kirchhoff):
  - ✓ Straight lines normal to the mid-surface remain straight after deformation
  - ✓ straight lines normal to the mid-surface remain normal to the mid-surface after deformation
  - ✓ the thickness of the plate does not change during deformation




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# Introduction

- Thin plates (Hyp. Kirchhoff):
  - ✓ According to these hypothesis, there is a neutral plane in the plate with no strains. Neutral plane is equivalent to neutral axis in beams
  - ✓ The strains field can be expressed as a function of a single variable, w (displacement of the mid-plane)
  - ✓ The problem has an exact solution in a limited number of cases. The partial differential equation must be degenerate in a ordinary differential equation
  - ✓ The problem is usually solved by numerical integration to find an approximated solution.

# Introduction

- Additional hypothesis:

  - ✓ Elastic-linear behaviour
  - ✓ Small deformations

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# Chapter 4. Plates and Shells

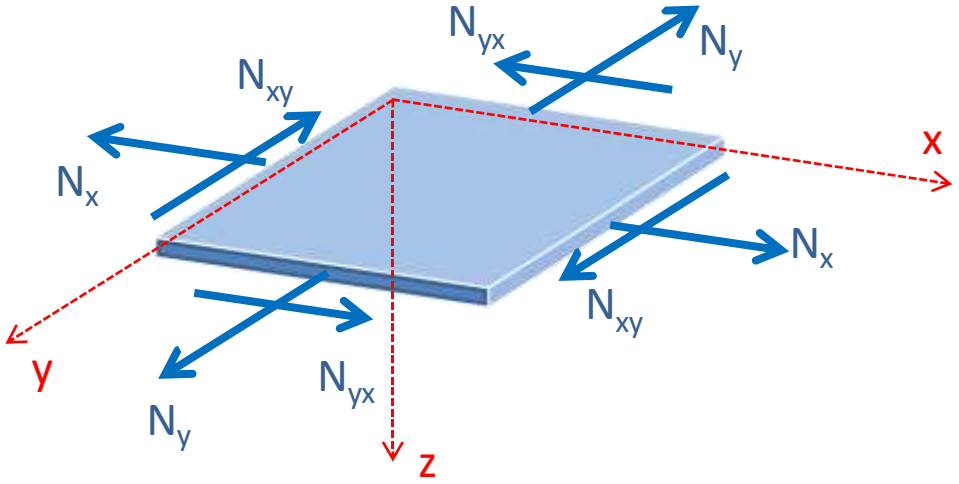
# Bending of thin plates

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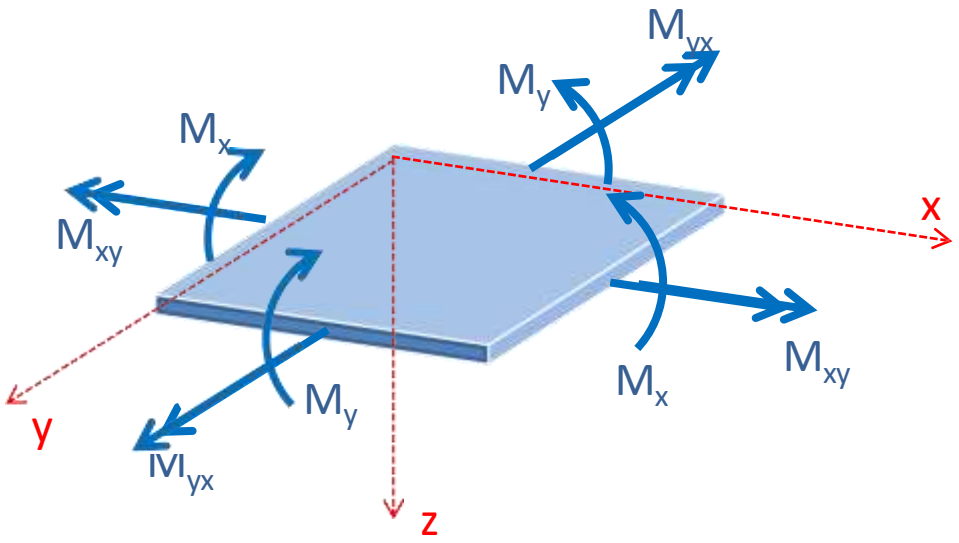
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# Equilibrium equations

- External forces:




- External moments:




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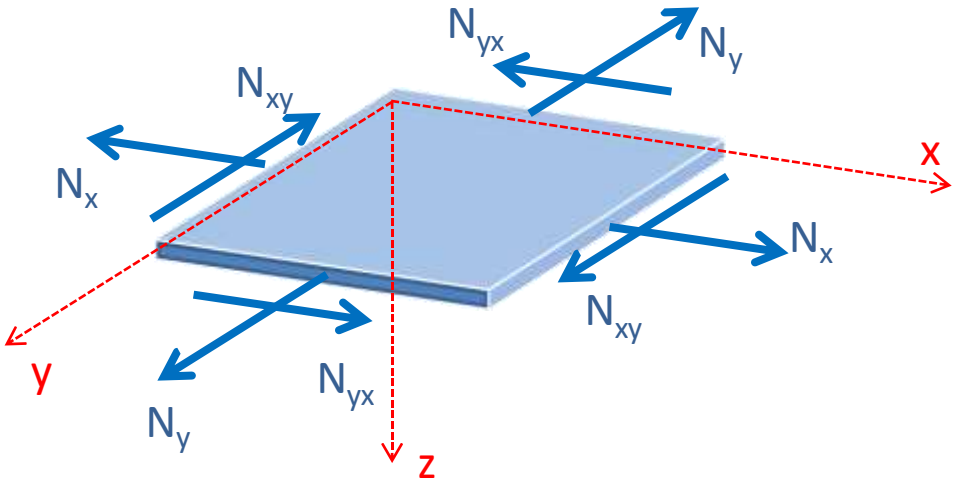
# Equilibrium equations

- External forces are obtained by the through-the-thickness integration of normal and shear stressess:

$$N_x = \int_z \sigma_{xx}\, dz$$

$$N_y = \int_z \sigma_{yy}\, dz$$

$$N_{xy} = \int_z \sigma_{xy}\, dz$$




External forces and external moments are defined by unit length

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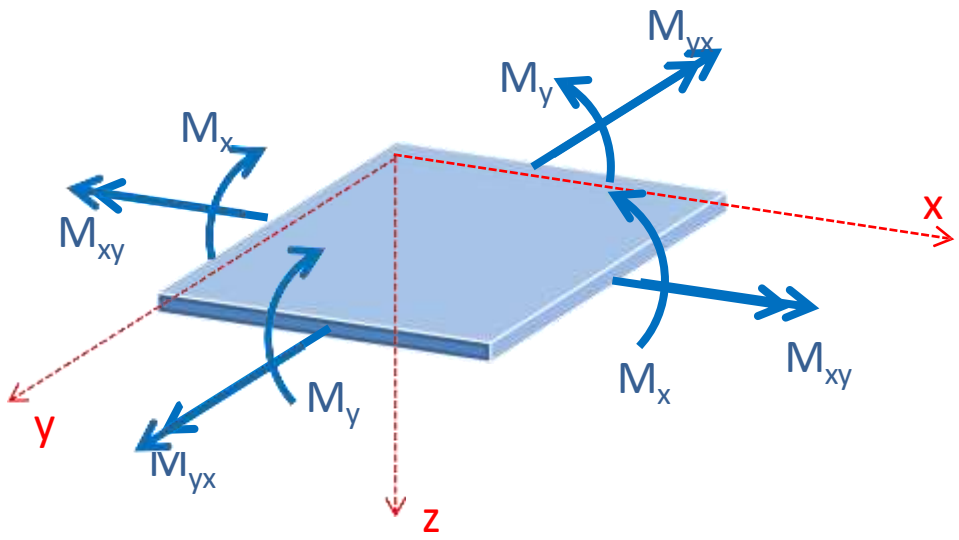
# Equilibrium equations

- External bending and torsion moments:

$$M_x = \int_z \sigma_{xx} z dz$$

$$M_y = \int_z \sigma_{yy} z dz$$

$$M_{xy} = \int_z \sigma_{xy} z dz$$



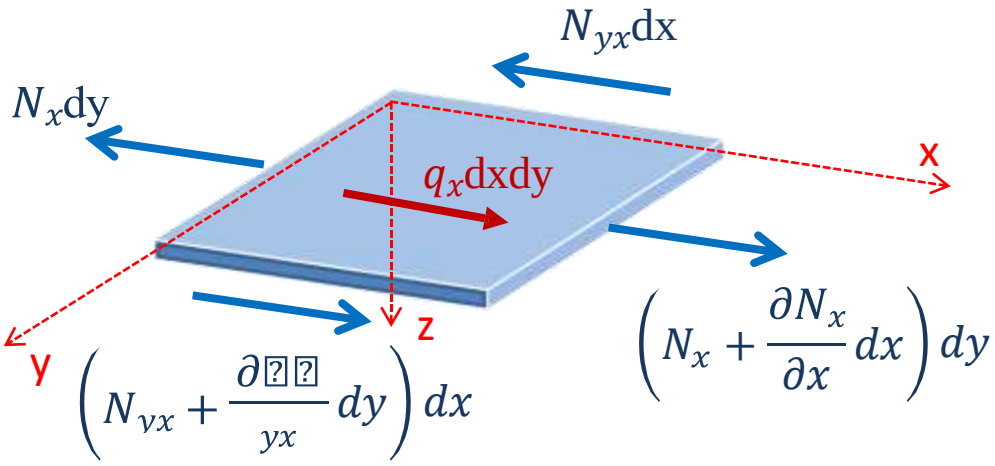

External forces and external moments are defined by unit length

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# Equilibrium equations

- Equilibrium in a plate differential element with dimensions: dx, dy

Equilibrium in x direction:



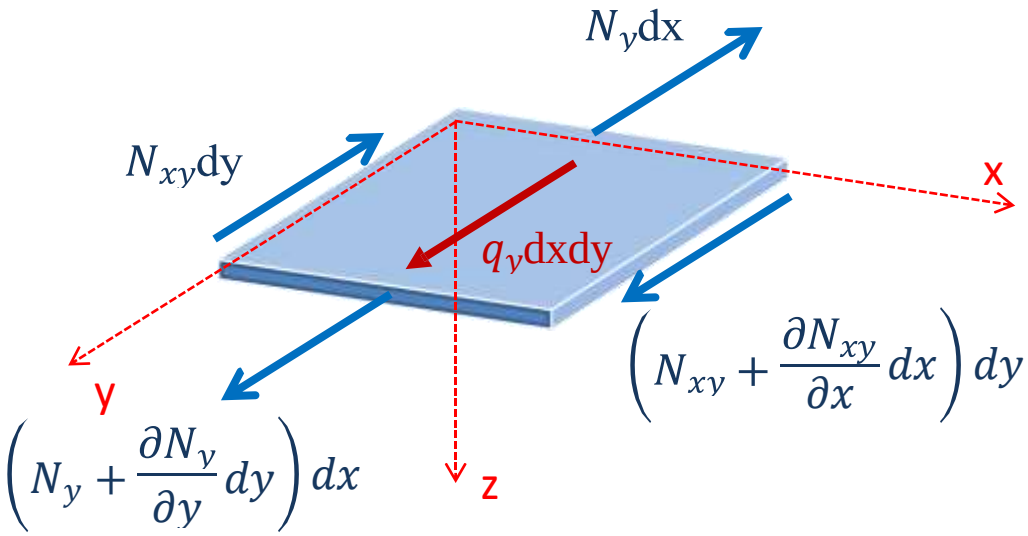

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = q_x$$

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# Equilibrium equations

- Equilibrium in a plate differential element with dimensions: dx, dy

Equilibrium in y direction:



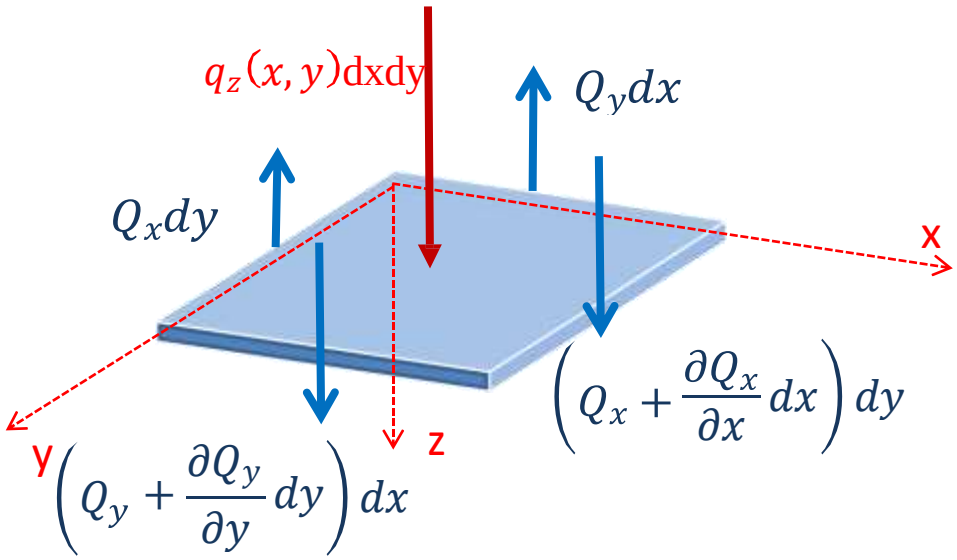

$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=q_y$$

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# Equilibrium equations

- Equilibrium in a plate differential element with dimensions: dx, dy

Equilibrium in z direction:



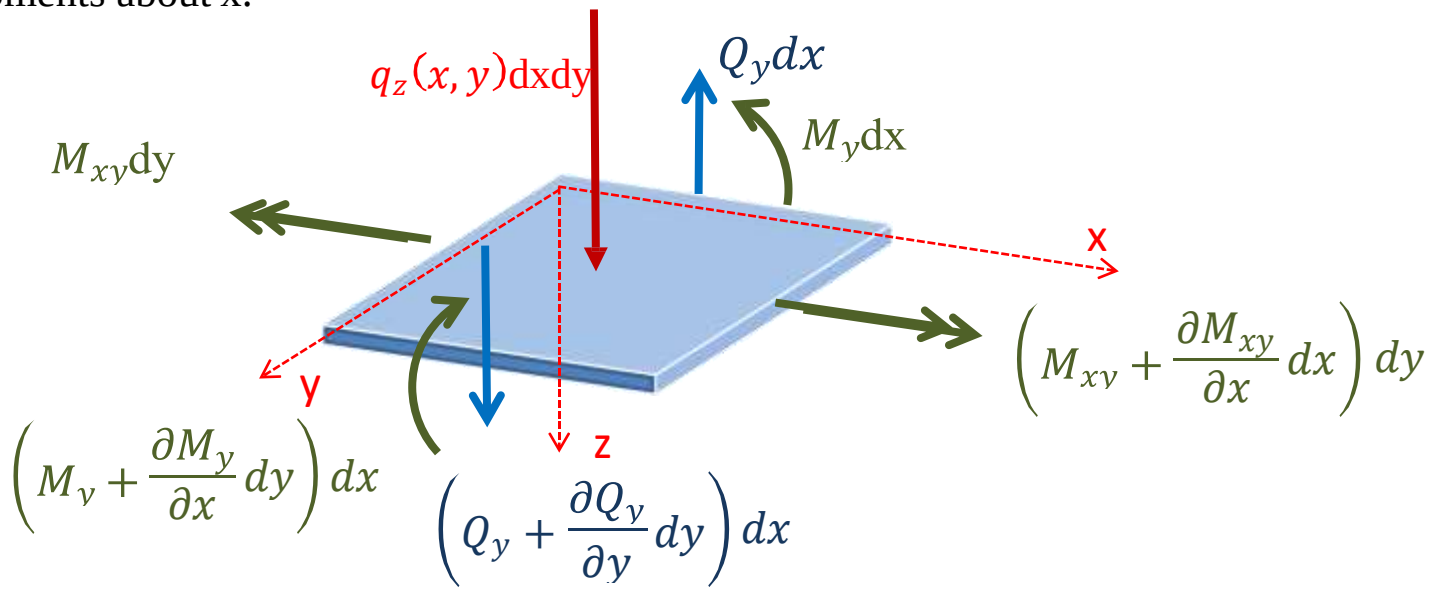

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q$$

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# Equilibrium equations

- Equilibrium in a plate differential element with dimensions: dx, dy

Equilibrium of moments about x:



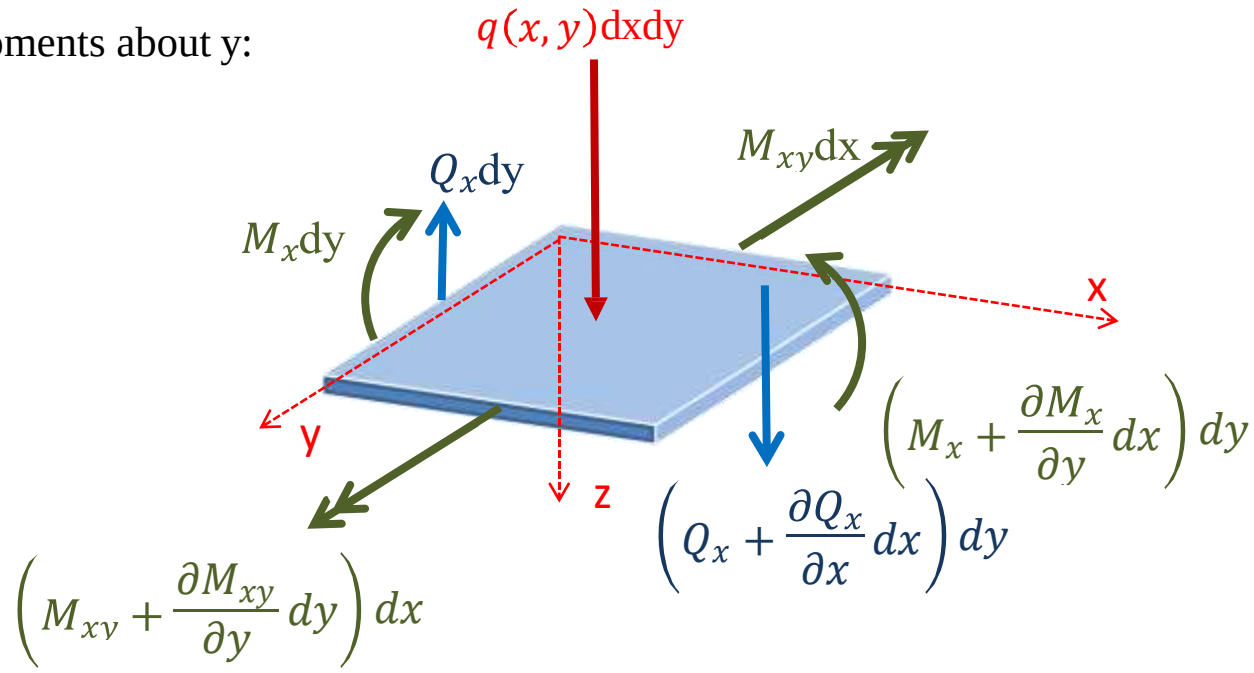

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = Q_y$$

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# Equilibrium equations

- Equilibrium in a plate differential element with dimensions: dx, dy

Equilibrium of moments about y:



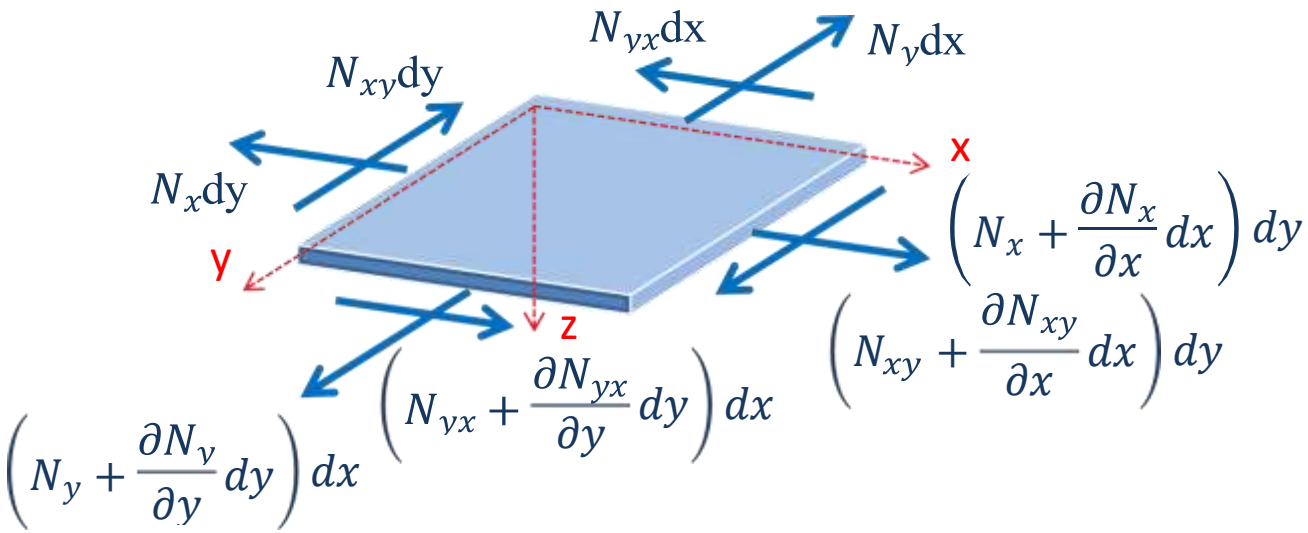

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}=Q_x$$

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# Equilibrium equations

- Equilibrium in a plate differential element with dimensions: dx, dy

Equilibrium of moments about z:




$$\sum M_z = 0 \longrightarrow \text{It is automatically satisfied}$$

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# Equilibrium equations

- Equilibrium in a plate differential element. Summary:

$$\sum F_x = 0 \rightarrow \frac{\partial N_y}{\partial x} + \frac{\partial N_{yx}}{\partial y} = q_x$$

$$\sum F_y = 0 \rightarrow \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = q_y$$

$$\sum F_z = 0 \rightarrow \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q$$

$$\sum M_x = 0 \rightarrow \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = Q_y$$

$$\sum M_y = 0 \rightarrow \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x$$

$$\sum M_z = 0 \rightarrow \text{Identity}$$

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# Equilibrium equations

- The problem of in-plane loads is mathematically uncoupled of the problem of out-of-plane loads.
- The combination of out-of-plane expression yields:

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}=Q_y \rightarrow \frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}=-q \leftarrow \frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}=Q_x$$

$$\downarrow$$

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial M_{xy}}{\partial x \partial y}+\frac{\partial^2 M_y}{\partial y^2}+q=0$$

Second order differential equation obtained from equilibrium equations

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Chapter 4. Plates and Shells

## Session 5. Bending of thin plates

1. Introduction
2. Equilibrium equations
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# Bending equations

A first order analysis of the strains yields:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{xz} = 2\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

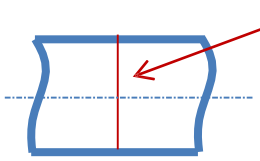
$$\gamma_{yz} = 2\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

Considering Kirchhoff hypothesis: $\longrightarrow$ $\varepsilon_{zz} = 0$

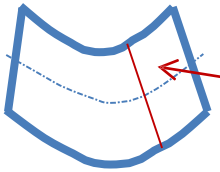
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# Bending equations

Non deformed Section

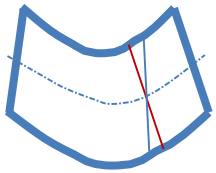


**Kirchhoff–Love plate theory**



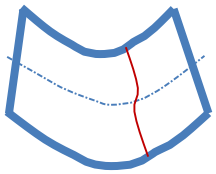
The section remains straight and perpendicular to the mid-plane

**Mindlin–Reissner theory(One-order theory)**



The section remains straight but not perpendicular to the mid-plane

The section does not remain straight



**Higher-order theories**

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# Bending equations

Kirchhoff hypothesis:

$$\varepsilon_{xx} = \frac{\partial u^0}{\partial x} + z\frac{\partial \theta_1}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v^0}{\partial y} + z\frac{\partial \theta_2}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} + z\left(\frac{\partial \theta_1}{\partial y} + \frac{\partial \theta_2}{\partial x}\right)$$

$$\gamma_{xz} = \theta_1 + \frac{\partial w}{\partial x}$$

$$\gamma_{yz} = \theta_2 + \frac{\partial w}{\partial y}$$

Neglecting displacement produced by shear strains (thin plate)

$$\theta_1 = -\frac{\partial w}{\partial x}$$

$$\theta_2 = -\frac{\partial w}{\partial y}$$

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# Bending equations

Relationship between stresses and strains:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 & 0 & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 & 0 & 0 \\ 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$$

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# Bending equations

The previous equation is multiplied by coordinate z and integrated through the thickness to obtain the relationship between moments and displacements

$$\begin{Bmatrix} \int \sigma_{xx} z dz \\ \int \sigma_{yy} z dz \\ \int \tau_{xy} z dz \\ \int \tau_{xz} z dz \\ \int \tau_{yz} z dz \end{Bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 & 0 & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 & 0 & 0 \\ 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \int \varepsilon_{xx} z dz \\ \int \varepsilon_{yy} z dz \\ \int \gamma_{xy} z dz \\ \int \gamma_{xz} z dz \\ \int \gamma_{yz} z dz \end{Bmatrix}$$

$$M_x = \frac{Et^3}{12\left(1-\nu^2\right)}\left(\frac{\partial \theta_1}{\partial x} + \nu \frac{\partial \theta_2}{\partial y}\right)$$

$$M_y = \frac{Et^3}{12\left(1-\nu^2\right)}\left(\nu \frac{\partial \theta_1}{\partial x} + \frac{\partial \theta_2}{\partial y}\right)$$

$$M_x = \frac{Gt^3}{12}\left(\frac{\partial \theta_1}{\partial y} + \nu \frac{\partial \theta_2}{\partial x}\right)$$

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# Bending equations

Neglecting the displacement produced by shear force (thin plate) and using the equilibrium equations it yields:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -D\left(\nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_x = -D(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q(x,y)$$

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

Fourth-order partial differential equation for a plate subjected to a transverse distributed load.

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# Bending equations

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

The parameter D is the equivalent bending stiffness of the plate

$$D = \frac{Eh^3}{12\left(1-\nu^2\right)}$$

- Boundary conditions are necessary to solve the differential equation

- There are different solutions for the displacement field, w(x,y), depending on boundary conditions

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# Chapter 4. Plates and Shells

## Bending of thin plates

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# Boundary conditions

- Clamped end: Displacements and rotations are restrained.

$$w(x_0) = 0$$
$$\frac{\partial w}{\partial x}(x_0) = 0$$
$$\frac{\partial w}{\partial y}(x_0) = 0$$

$$\rightarrow \quad \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

- Pinned end: Displacements and bending moments are restrained.

$$w(x_0) = 0$$
$$M_{nn}(x_0) = 0$$

$$\rightarrow \quad \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

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# Boundary conditions

- Free end: Bending moment, torsion moment and shear force are restrained.

$$\begin{aligned} M_{nn}(x_0) &= 0 \\ M_{nt}(x_0) &= 0 \\ Q_n(x_0) &= 0 \end{aligned} \longrightarrow \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

- Symmetrical axis: This is not a boundary condition but it is usual in the analysis of plates. Supposing that y is a symmetrical axis:

$$\begin{aligned} M_{xy}(x_0) &= 0 \\ Q_y(x_0) &= 0 \\ \frac{\partial w}{\partial y}(x_0) &= 0 \end{aligned} \longrightarrow \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

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# Chapter 4. Plates and Shells

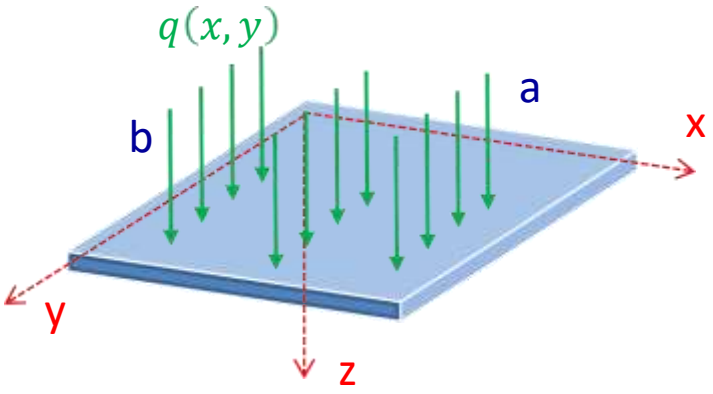
# Bending of thin plates

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# Example

$q(x,y)\mathrm{dxdy}$




- Plate dimensions: a·b
- Pinned at all the ends

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

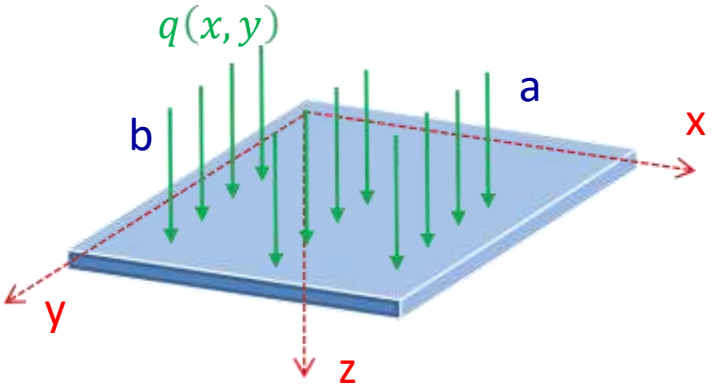
Navier solution:

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$A_{mn}$ are the coefficients that must satisfy the differential equation

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# Example




- Plate dimensions: a·b
- Pinned at all the ends

$$\frac{\partial^4 w}{dx^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

External load q(x,y) can also be expressed as a function of Fourier series:

$$q(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \longrightarrow A_{mn} = \frac{1}{\pi^4 D\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2} a_{mn}$$

$$w(x,y) = \frac{1}{\pi^4 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_{mn}}{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2} \sin\frac{m\pi x}{a} \sin\frac{m\pi y}{b}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Chapter 4. Plates and Shells
# Bending of thin plates

1. Introduction
2. Equilibrium equations
3. Bending equations
4. Boundary conditions
5. Example
6. References

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# References


1. **Megson, T.H.G. "Aircraft Structures"**
   **Elsevier. 2007**
   Cap.7 Bending of thin plates

2. **Timoshenko, Stephen "Theory of plates and shells "**
   **McGraw-Hill, 1959**

1. **Monleón Cremades, Salvador "Análisis de vigas, arcos, placas y láminas : una presentación unificada"**
   **Universidad Politécnica de Valencia. 1999**

2. **J. A. Jurado Albarracín-Martinón y S. Hernández Ibáñez, "Análisis estructural de placas y láminas". Tercera Edición. Andavira editora 2014**


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# Aerospace Structures

## Chapter 4. Plates and Shells

### Circular plates

# Chapter 4. Plates and Shells

## Circular plates

1. Equilibrium equations in polar coordinate system
2. Bending equations in polar coordinate system
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4. Examples
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# Equilibrium equations

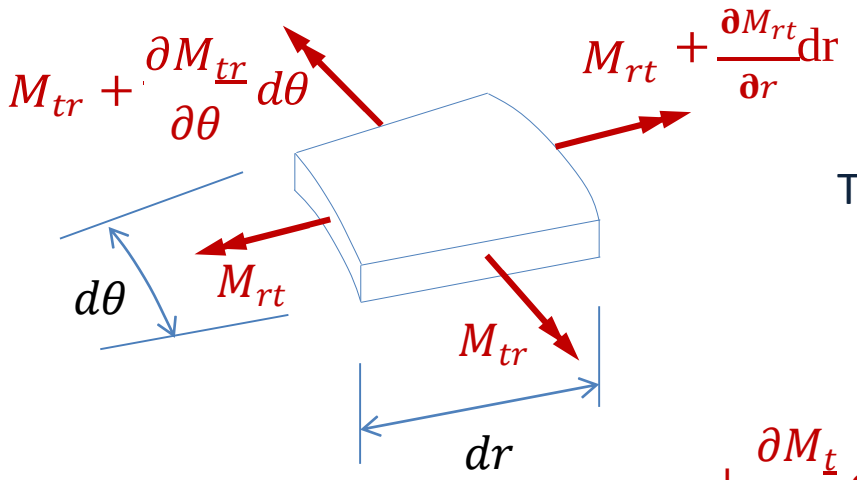
- Bending and torsion moments:

Defined per unit length

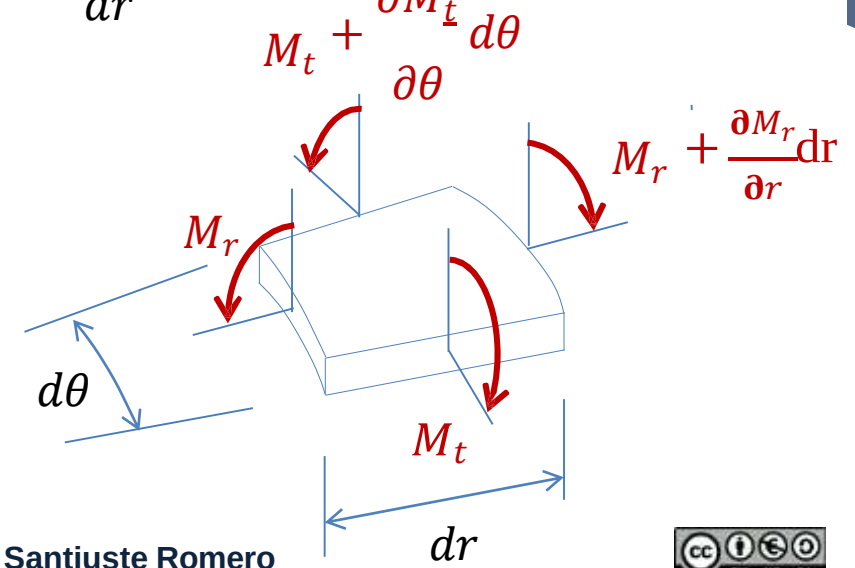
$$M_r = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{rr} \cdot z dz$$

$$M_t = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{\theta\theta} \cdot z \cdot dz$$

$$M_{rt} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{r\theta} \cdot z \cdot dz$$




Torsion moment




Bending moments

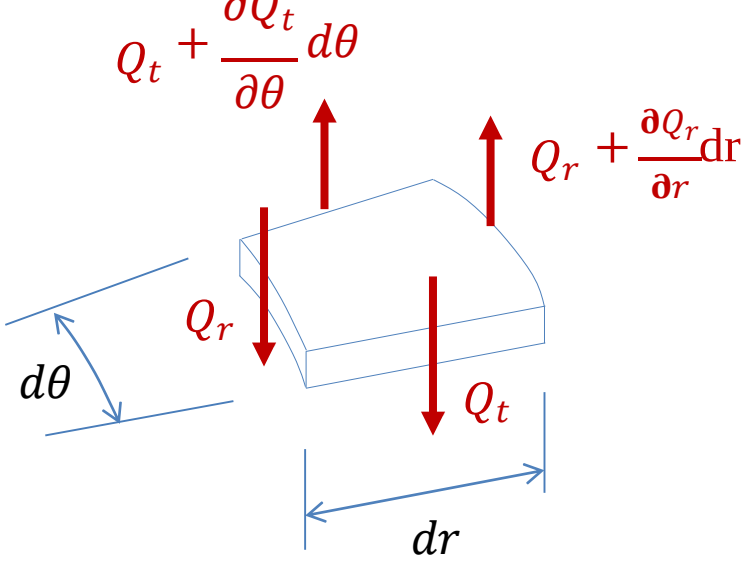
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# Equilibrium equations

- Shear force

$$Q_r = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{rz} \cdot z dz$$

$$Q_t = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{\theta z} \cdot z dz$$




Defined per unit length

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# Equilibrium equations

Internal forces can be expressed in terms of vertical displacement (same procedure than in Cartesian coordinate system)

$$M_r = -D \cdot \left( \frac{\partial^2 w}{\partial r^2} + \nu \cdot \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right)$$

$$M_t = -D \cdot \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \nu \frac{\partial^2 w}{\partial r^2} \right)$$

$$M_{rt} = -D \cdot (1-\nu) \left( \frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial w}{\partial \theta} \right)$$

$$Q_r = -D \cdot \frac{\partial}{\partial r} \left( \nabla^2 w \right)$$

$$Q_t = -\frac{D}{r} \cdot \frac{\partial}{\partial \theta} \left( \nabla^2 w \right)$$

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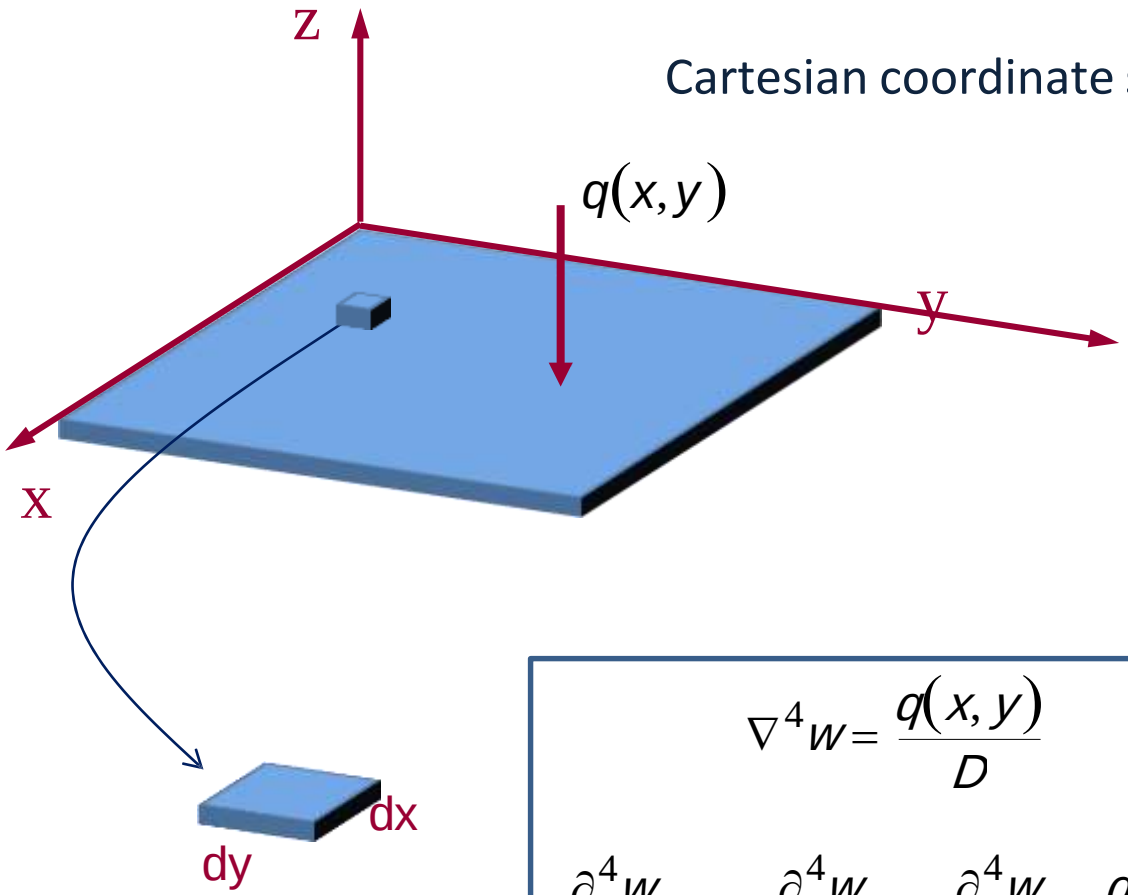
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# Bending equations




Cartesian coordinate system:

Del operator

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

Laplace operator (in-plane)

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla^4 w = \frac{q(x,y)}{D}$$

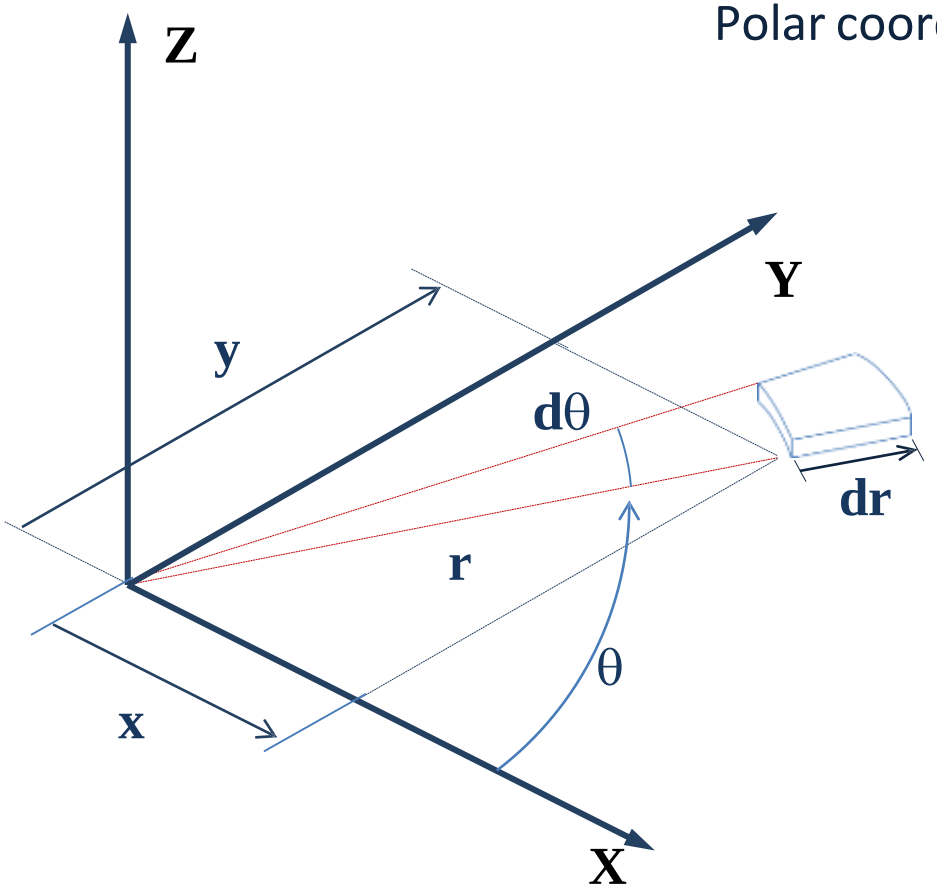
$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D}$$

$$D = \frac{E \cdot h^3}{12 \cdot \left(1 - \nu^2\right)}$$

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# Bending equations

Polar coordinate system:




Del operator

$$\nabla = \frac{\partial}{\partial r}\vec{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\vec{\theta} + \frac{\partial}{\partial z}\vec{u}_z$$

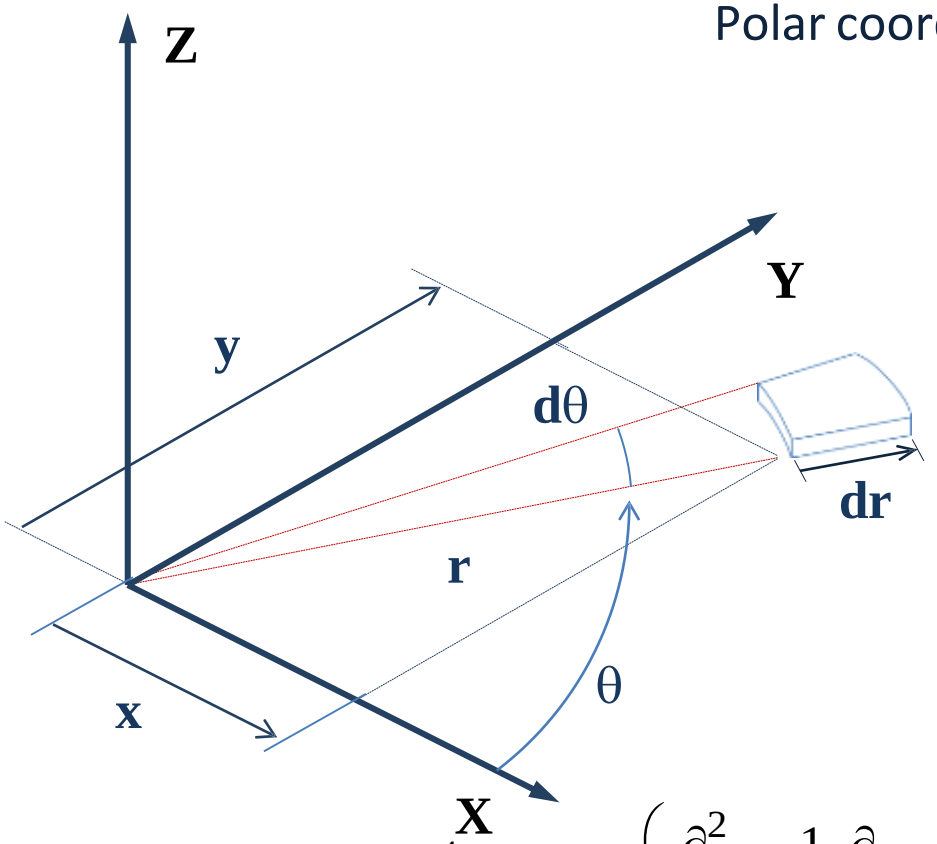
Laplace operator (in-plane)

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

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# Bending equations

Polar coordinate system:




$$\nabla^4 w = \frac{q}{D}$$

$$\nabla^4 w(r,\theta) = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) w = \frac{q(r,\theta)}{D}$$

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# Bending equations

## Solution of the differential equation

$$w = w_h + w_p$$

$$\nabla^4 w_h = 0 \qquad \nabla^4 w_p = \frac{q}{D}$$

The solution of the homogeneous equation can be expressed as the next series:

$$w_h(r,\theta) = \sum_{m=0}^{\infty} R_m(r)\cdot\cos(m\cdot\theta) + \sum_{m=1}^{\infty} S_m(r)\cdot sen(m\cdot\theta)$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right) R_m = 0$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right) S_m = 0$$

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# Bending equations

The solutions to these equations are:

$$R_o = A_o + B_o \cdot \ln r + C_o \cdot r^2 + D_o \cdot r^2 \cdot \ln r$$

$$R_1, S_1 = A_1 \cdot r + B_1 \cdot \frac{1}{r} + C_1 \cdot r^3 + D_1 \cdot r \cdot \ln r$$

$$R_m, S_m = A_m \cdot r^m + B_m \cdot \frac{1}{r^m} + C_m \cdot r^{m+2} + D_m \cdot \frac{1}{r^{m-2}}$$

The coefficient of these equations can be found imposing boundary conditions

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# Chapter 4. Plates and Shells

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# Boundary conditions

## Clamped in r = a

$$w\big|_{r=a} = 0$$

$$\frac{\partial w}{\partial r}\Big|_{r=a} = 0$$

## Simply supported in r = a

$$w\big|_{r=a} = 0$$

$$M_r\big|_{r=a} = 0$$

## Free end in r = a

$$\left(Q_r + \frac{1}{r}\cdot\frac{\partial M_{tr}}{\partial \theta}\right)\Big|_{r=a} = 0$$

$$M_r\big|_{r=a} = 0$$

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# Chapter 4. Plates and Shells

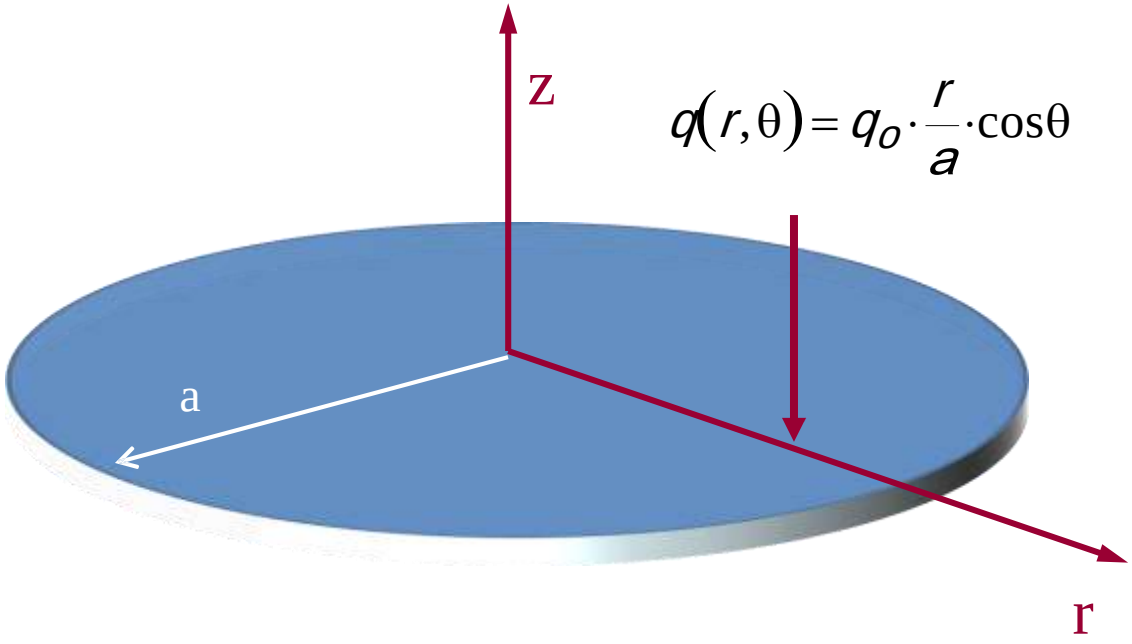
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# Examples

## Example: Load directly related to radius



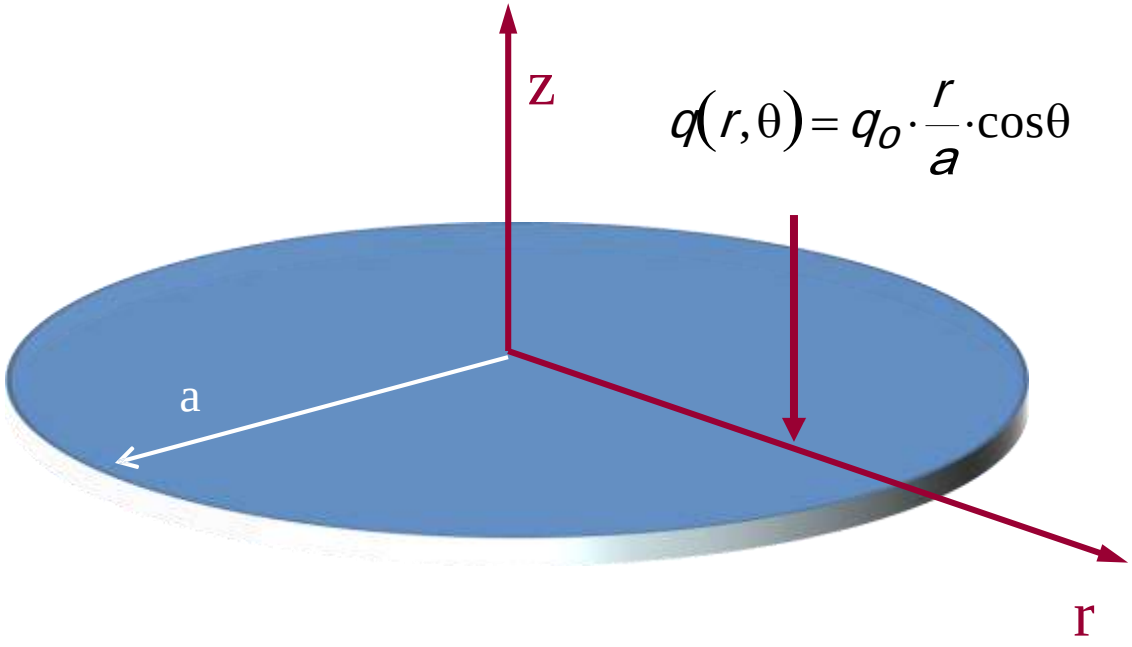

$$q(r,\theta) = q_o \cdot \frac{r}{a} \cdot \cos\theta$$

$$\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) w = \frac{q_o \cdot r \cdot \cos\theta}{a \cdot D}$$

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# Examples

## Example: Load directly related to radius




$$q(r,\theta) = q_o \cdot \frac{r}{a} \cdot \cos\theta$$

Particular solution

$$\nabla^4 w_p = \frac{q_o \cdot r \cdot \cos\theta}{a \cdot D}$$

There are fourth derivatives in the equation, thus this is a possible solution:

Where A is a constant

$$w_p(r,\theta) = A \cdot r^5 \cdot \cos\theta$$

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# Examples

## Example: Load directly related to radius

$$w_p(r,\theta) = A \cdot r^5 \cdot \cos\theta$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) w = \frac{q_o \cdot r \cdot \cos\theta}{a \cdot D}$$

$$w_p(r,\theta) = 5.2 \cdot 10^{-3} \cdot \frac{q_o \cdot r^5 \cdot \cos\theta}{a \cdot D}$$

Particular solution

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# Examples

**Example: Load directly related to radius**

Homogenous solution

$$w_h(r,\theta) = \sum_{m=0}^{\infty} R_m(r)\cdot\cos(m\cdot\theta) + \sum_{m=1}^{\infty} S_m(r)\cdot sen(m\cdot\theta)$$

Since load only has terms in cosine, the solution is:

$$w_h(r,\theta) = R_1(r)\cdot\cos\theta$$

$$R_1(r) = A_1\cdot r + B_1\cdot\frac{1}{r} + C_1\cdot r^3 + D_1\cdot r\cdot\ln r$$

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# Examples

## Example: Load directly related to radius

Final solution:

$$w(r,\theta) = \frac{q_o \cdot a^4}{192 \cdot D}\left(\rho^5 + A^* \cdot \rho + B^* \cdot \frac{1}{\rho} + C^* \cdot \rho^3 + D^* \cdot \rho \cdot \ln \rho\right) \cdot \cos\theta \qquad \rho = \frac{r}{a}$$

Constant are calculated imposing boundary conditions

- Vertical displacement is null in the midpoint:

$$B^* = 0$$

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# Examples

## Example: Load directly related to radius

$$M_r = -D \cdot \left( \frac{\partial^2 w}{\partial r^2} + \nu \cdot \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right)$$

$$M_t = -D \cdot \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \nu \frac{\partial^2 w}{\partial r^2} \right)$$

$$M_{rt} = -D \cdot (1-\nu) \left( \frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial w}{\partial \theta} \right)$$

$$M_r = -D \cdot \frac{q_o \cdot a^2}{192} \left[ (20 + 4 \cdot \nu) \cdot \rho^3 + (6 + 2 \cdot \nu) \cdot C^* \cdot \rho + \frac{D^* \cdot (1+\nu)}{\rho} \right] \cdot \cos\theta$$

$D^* = 0$ Otherwise bending moment is infinite in the midpoint

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# Examples

## Example: Load directly related to radius

$$w(r,\theta) = \frac{q_o \cdot a^4}{192 \cdot D}\left( \frac{r}{a^5}^5 + A^* \cdot \frac{r}{a} + C^* \cdot \frac{r^3}{a^3} \right) \cdot \cos\theta$$

The other constants are found imposing boundary conditions at the external edge r=a

Example, simply supported plate:

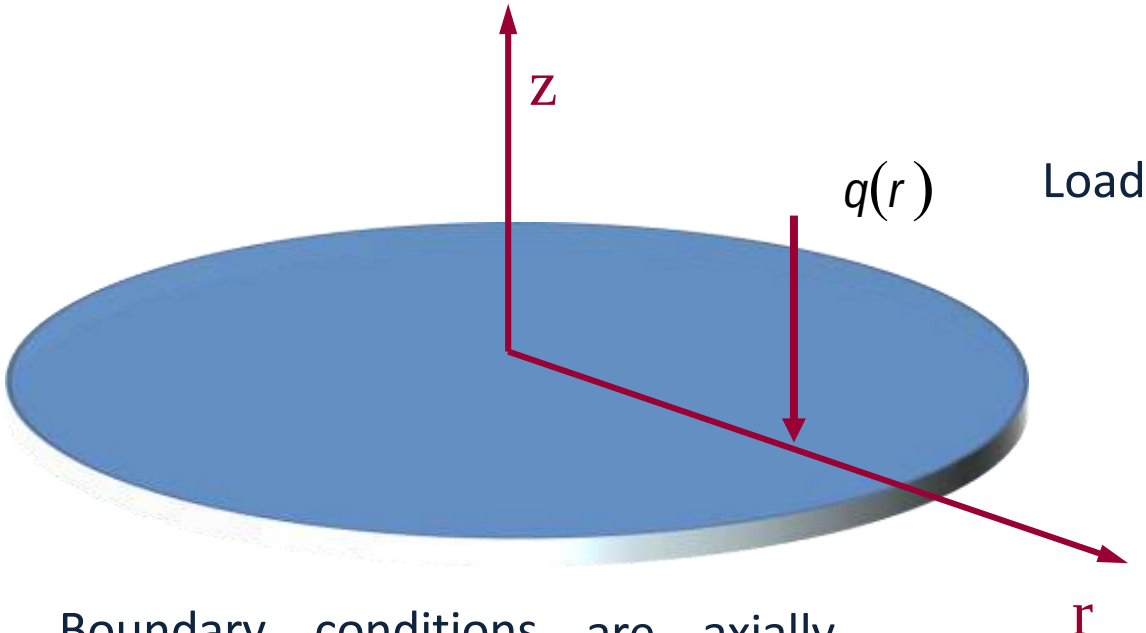
$$w\big|_{r=a} = 0$$
$$M_r\big|_{r=a} = 0$$

$$\Rightarrow \quad A^* = \frac{7+\nu}{3+\nu}$$
$$C^* = -2\frac{5+\nu}{3+\nu}$$

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# Examples

## Example: Axial symmetry




Loads are angle independent

Boundary conditions are axially symmetrical

$$\nabla^4 w = \frac{q}{D}$$

$$\nabla^4 w(r) = \left( \frac{d^2}{dr^2} + \frac{1}{r}\frac{\partial}{\partial r} \right)\left( \frac{d^2}{dr^2} + \frac{1}{r}\frac{\partial}{dr} \right) = \frac{q(r)}{D}$$

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# Examples

## Example: Axial symmetry

Internal forces equations considering angle independency

$$M_r = -D \cdot \left( \frac{d^2 w}{dr^2} + \nu \frac{1}{r} \frac{dw}{dr} \right)$$

$$M_t = -D \cdot \left( \nu \frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right)$$

$$M_{rt} = 0$$

$$Q_r = -D \cdot \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} \left( r \frac{dw}{dr} \right) \right)$$

$$Q_t = 0$$

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# Examples

## Example: Axial symmetry

Differential equation can be integrated when load is uniform:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)=\frac{q_o}{D}$$

$$w_h(r)=A_o+B_o\cdot\ln r+C_o\cdot r^2+D_o\cdot r^2\cdot\ln r$$

$$w_p(r)=\frac{q_o\cdot r^4}{64D}$$

Coefficients are found imposing boundary conditions

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# References


1. **Megson, T.H.G. "Aircraft Structures"**
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## Department of Continuum Mechanics and Structural Engineering

# Aerospace Structures

## Chapter 4. Plates and Shells

### Shells of revolution

**Authors:** Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero



# Chapter 4. Plates and Shells

## Shells of revolution

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2. Membrane theory
3. Basic geometrical relations
4. Equilibrium equations for shells of revolution
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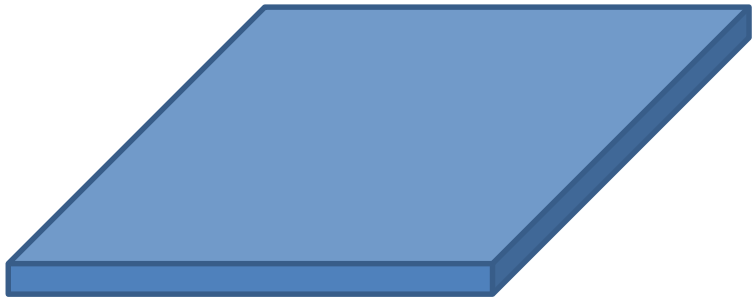
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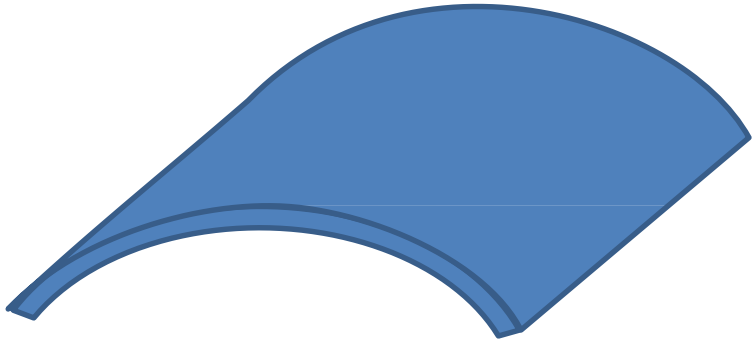
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# Introduction

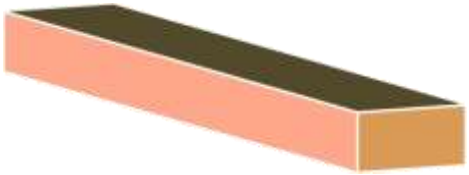
Differences between plates and shells?



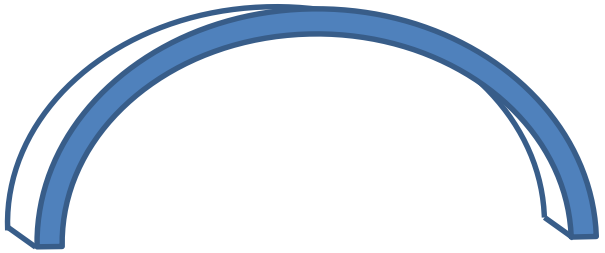
Plate



Shell



Beam



Arch

Differences between beams and arches?

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## General definition

A structural membrane or shell is a curved surface structure. Usually, it is capable of transmitting loads in more than two directions to supports. It is highly efficient structurally when it is so shaped, proportioned, and supported that it transmits the loads without bending or twisting.

A membrane or a shell is defined by its middle surface, halfway between its extrados, or outer surface and intrados, or inner surface. Thus, depending on the geometry of the middle surface, it might be a type of dome, barrel arch, cone, or hyperbolic paraboloid. Its thickness is the distance, normal to the middle surface, between extrados and intrados.

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## Advantages of shells structures

1. Efficiency of load-carrying behaviour
2. High degree of reserved strength and structural integrity
3. High specific strength (strength/ weight ratio)
4. Very high stiffness
5. Containment of space

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# Introduction

## Thin shells

A thin shell is a shell with a thickness relatively small compared with its other dimensions. But it should not be so thin that deformations would be large compared with the thickness.

Calculation of the stresses in a thin shell generally is carried out in two major steps, both usually involving the solution of differential equations. In the first, bending and torsion are neglected (membrane theory). In the second step, corrections are made to the previous solution by superimposing the bending and shear stresses that are necessary to satisfy boundary conditions (bending theory)

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## Examples of thin shells

- Civil and architectural engineering
  - ✓ Large span-roofs
  - ✓ Liquid-retaining structures and water tanks
  - ✓ Concrete arch domes
- Mechanical engineering
  - ✓ Piping systems
  - ✓ Turbine disks
  - ✓ Pressure vessels
- Biomechanics

AEROSPACE STRUCTURES

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## Kirchhoff-Love hypothesis

- Thickness is small comparing to the curvature radius of the mid-surface
- Small displacements and strains (equilibrium is verified in the undeformed shape)
- Straight lines normal to the mid-surface remain straight and normal after deformation
- Stresses in perpendicular direction to the mid-surface are neglected

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Chapter 4. Plates and Shells

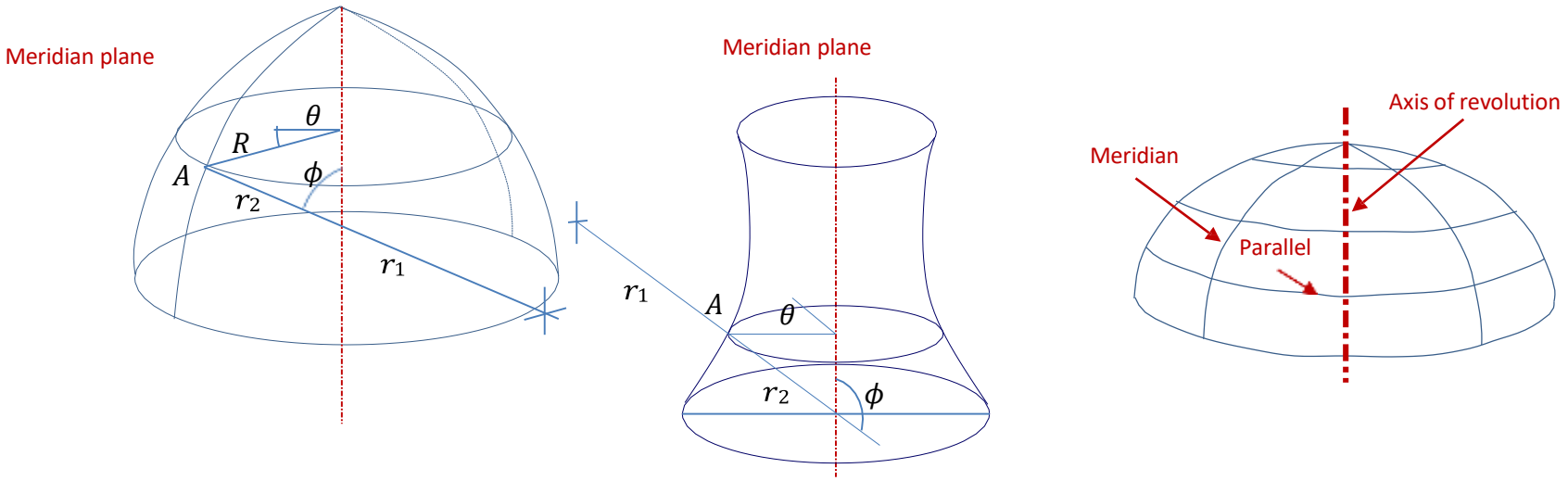
## Shells of revolution

1. Introduction
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**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

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# Geometrical relations




$\phi$: meridian angle

$r_1$: first main radius. It is the curvature radius of the meridian

$\theta$: circumferential angle

$r_2$: second main radius. It is the distance to the revolution axis in normal direction to the shell surface

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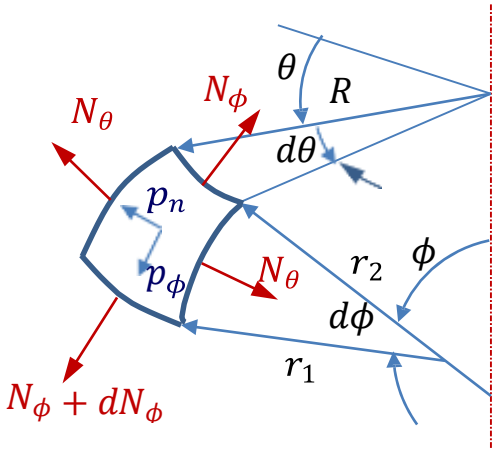
# Chapter 4. Plates and Shells

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# Equilibrium equations




- Bending and torsion moments are neglected (Membrane theory)
- Shear forces are neglected in mid-surface (Axial-symmetry)

## Internal forces:

- $N_\phi$: per unit length in meridian direction
- $N_\theta$: per unit length in circumferential direction (independent on $\theta$)

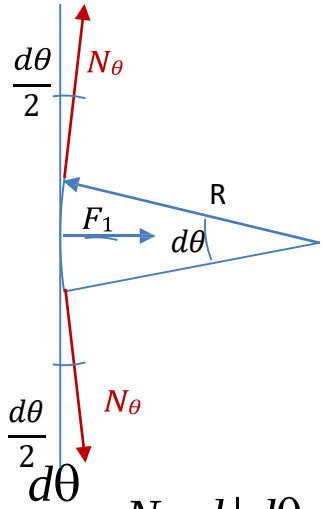
## External loads per unit surface

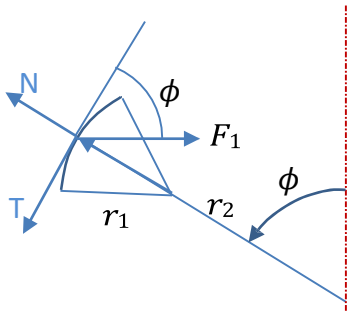
- $p_n$: in normal direction
- $p_\phi$: in meridian direction
- $p_\theta$: does not exist (axial-symmetrical load)

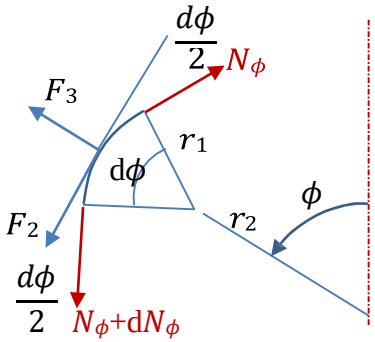
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# Equilibrium equations

## Equilibrium in meridian direction




$$F_1 = 2N_{\theta 1}\, rd\phi sen\frac{d\theta}{2} \approx N_{\theta 1}\, rd\phi d\theta$$

Tangential component

$$-F_1\cos\phi = -N_\theta r_1 \cos\phi d\phi d\theta$$

$$F_2 \approx N_\phi\, dRd\theta + RdN_\phi\, d\theta$$

$$F_2 \approx d\left(RN_\phi\right)d\theta$$

Tangential component of the external load:

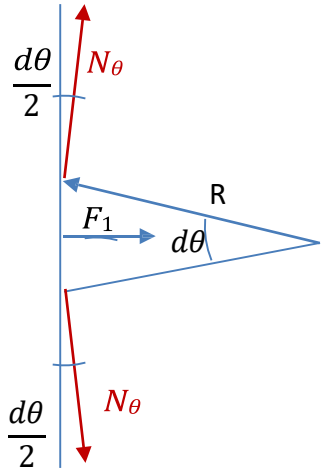
$$p_\phi r_1 Rd\phi d\theta$$

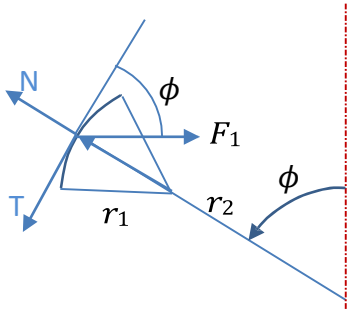
$$d\left(RN_\phi\right) - N_\theta r_1 \cos\phi d\phi + p_\phi r_1 Rd\phi = 0$$

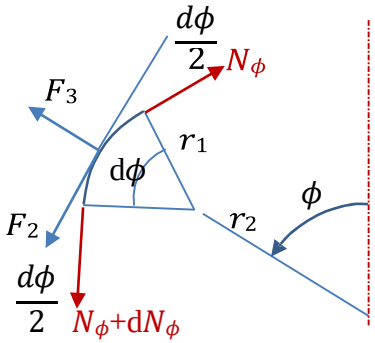
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# Equilibrium equations

## Equilibrium in normal direction




$$F_1 \approx N_\theta\, r d\phi d\theta$$

$$F_3 = -N_\phi\, R d\theta sen\frac{d\phi}{2} - \left(N_\phi + dN_\phi\right)(R+dr)d\theta sen\frac{d\phi}{2}$$

Normal component:

$$-F_1 sen\phi = -N_\theta\, r sen\phi d\phi d\theta$$

$$F_3 \approx -RN_\phi d\theta d\phi$$

Normal component of the external load:

$$p_n r_1 R d\phi d\theta$$

$$\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} = p_n$$

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# Equilibrium equations

$$d\left(RN_\phi\right) - N_\theta r_1 \cos\phi\, d\phi + p_\phi r_1 R d\phi = 0$$

$$\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} = p_n$$

$$d\left(RN_\phi\right) - r_1\left(r_2 p_n - \frac{r_2}{r_1} N_\phi\right)\cos\phi\, d\phi = -r_1 R p_\phi\, d\phi$$

Multiplying by $\operatorname{sen}\phi / d\phi$ and considering $R = r_2 \operatorname{sen}\phi$

$$\frac{d\left(r_2 \operatorname{sen}^2\phi N_\phi\right)}{d\phi} = r_1 r_2 p_n \cos\phi \operatorname{sen}\phi - r_1 r_2 \operatorname{sen}^2\phi\, p_\phi$$

Integrating over $\phi$

$$N_\phi = \frac{\int r_1 r_2 \operatorname{sen}\phi\left(p_n \cos\phi - p_\phi \operatorname{sen}\phi\right) d\phi + k}{r_2 \operatorname{sen}^2\phi}$$

k is obtained from boundary conditions

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# Chapter 4. Plates and Shells

## Shells of revolution

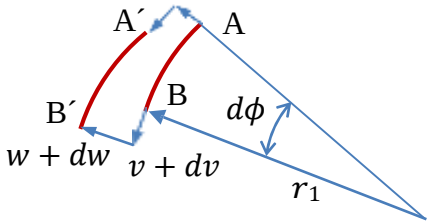
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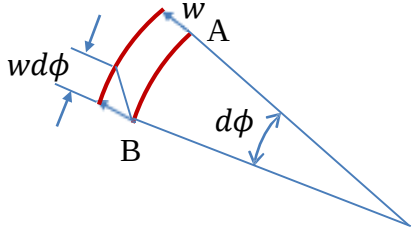
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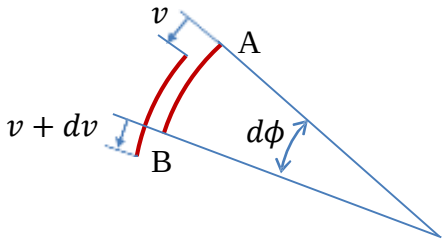
# Displacement equations

## Displacements: v, tangent to meridian; w, in normal direction

### Strains

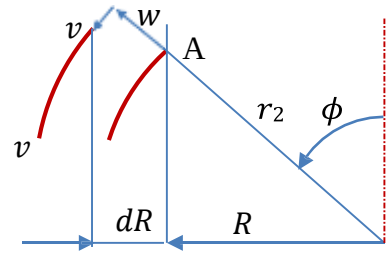



$$\varepsilon_\phi = \frac{A'B' - AB}{AB} = \frac{dv + wd\phi}{r_1 d\phi} = \frac{1}{r_1}\left(\frac{dv}{d\phi} + w\right)$$

v




$$\varepsilon_\theta = \frac{2\pi(R + dR) - 2\pi R}{2\pi R} = \frac{dR}{R} \qquad dR = v\cos\phi + w\,sen\phi$$

$$\varepsilon_\theta = \frac{v\cos\phi + w\,sen\phi}{R} = \frac{1}{r_2}\left(v\cot\phi + w\right)$$

with $R = r_2 sen\phi$

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# Displacement equations

## Strains

$$\frac{dv}{d\phi} + w = r\varepsilon_1 \qquad v\cot\phi + w = r\varepsilon_2$$

$$\frac{dv}{d\phi} - v\cot\phi = r\varepsilon_1 - r\varepsilon_2$$

## Hook laws

$$\varepsilon_\phi = \frac{1}{E}(\sigma_\phi - \nu\sigma_\theta) = \frac{1}{Eh}(N_\phi - \nu N_\theta)$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_\phi) = \frac{1}{Eh}(N_\theta - \nu N_\phi)$$

$$\frac{dv}{d\phi} - v\cot\phi = \frac{1}{Eh}\left[(r_1 + \nu r_2)N_\phi - (r_2 + \nu r_1)N_\theta\right]$$

$$\operatorname{sen}\phi\,\frac{d}{d\phi}\left(\frac{v}{\operatorname{sen}\phi}\right)$$

$$v = \operatorname{sen}\phi\left\{\frac{1}{Eh}\int\frac{1}{\operatorname{sen}\phi}\left[(r_1 + \nu r_2)N_\phi - (r_2 + \nu r_1)N_\theta\right]d\phi + k\right\}$$

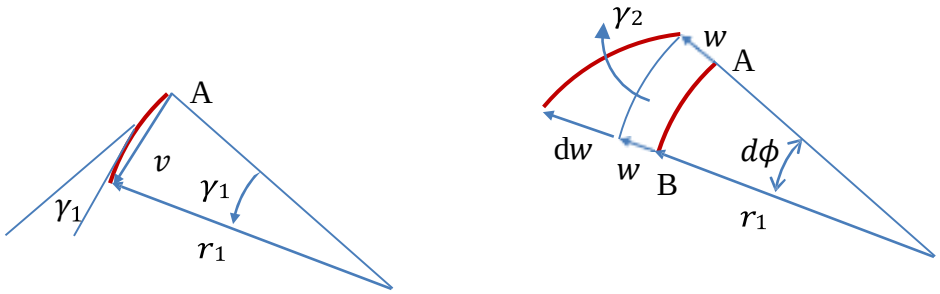
$$w = r_2\varepsilon_\theta - v\cot\phi = \frac{r_2}{Eh}(N_\theta - \nu N_\phi) - v\cot\phi$$

k is obtained from boundary conditions

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# Displacement equations

## Meridian rotation




$\gamma_1$: rotation motivated by the displacement v of point A

$\gamma_2$: rotation motivated by the difference between displacements w of points A and B

$\gamma$ : meridian rotation

$$\gamma = \gamma_1 - \gamma_2 = \frac{v}{r_1} - \frac{dw}{r_1 d\phi}$$

$\delta$ : horizontal displacement

$$\delta = R\varepsilon_\theta = \frac{1}{Eh} r_2 \operatorname{sen}\phi \left( N_\theta - \nu N_\phi \right)$$

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# Displacement equations

## Meridian rotation

$$\gamma = \gamma_1 - \gamma_2 = \frac{v}{r_1} - \frac{dw}{r_1 d\phi}$$

$$\frac{d}{d\phi}(v\cot\phi + w) = \frac{dv}{d\phi}\cot\phi - \frac{v}{sen^2\phi} = r_2\frac{d\varepsilon_\theta}{d\phi}$$

On the other hand

$$\left(\frac{dv}{d\phi} - v\cot\phi\right)\cot\phi = \frac{dv}{d\phi}\cot\phi - \frac{v\cos^2\phi}{sen^2\phi} = \frac{\cot\phi}{Eh}\left[(r_1 + \nu r_2)N_\phi - (r_2 + \nu r_1)N_\theta\right]$$

Subtracting the above equations and dividing by $r_1$

$$\gamma = \frac{1}{r_1}\left\{\frac{\cot\phi}{Eh}\left[(r_1 + \nu r_2)N_\phi - (r_2 + \nu r_1)N_\theta\right] - \frac{d}{d\phi}\left[\frac{1}{Eh}(N_\theta - \nu N_\phi)\right]\right\}$$

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# Shells of revolution

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# References


1. **J. A. Jurado Albarracín-Martinón y S. Hernández Ibáñez, "Análisis estructural de placas y láminas". Tercera Edición. Andavira editora 2014**
   Cap.7

2. **Timoshenko, Stephen "Theory of plates and shells " McGraw-Hill, 1959**
   Cap.14,15,16

3. **Ugural, A.C. "Stresses in beams, plates and shells". CRC. Taylor & Francis, 2009**


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# Aerospace Structures

## Chapter 4. Laminate and sandwich structures

### Laminate theory

# Chapter 4. Laminate and Sandwich Structures

## Laminate theory

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2. Anisotropic behaviour
3. Lamina stiffness matrix
4. Hypotheses
5. Laminate kinematics
6. Internal forces
7. Laminate stiffness matrix
8. Laminate configurations
9. Laminate equivalent constants
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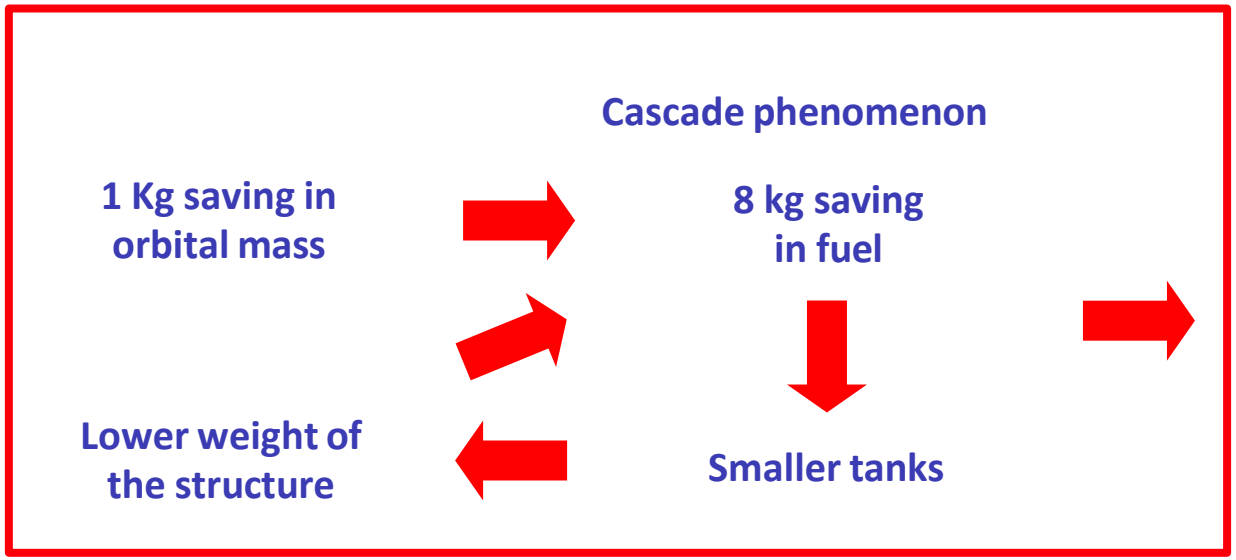
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# Introduction

## Material requirements in aerospace applications

- High strength and stiffness (stiffness is most important)
- Low thermal expansion coefficient (extreme temperatures -200/200ºC)
- Low density (essential)



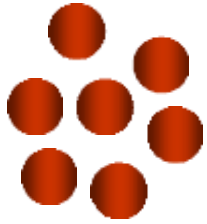

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# Introduction

## Composite definition

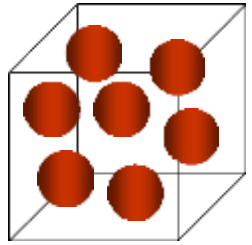


Matrix



Reinforcement

=



Composite material

Characteristics:

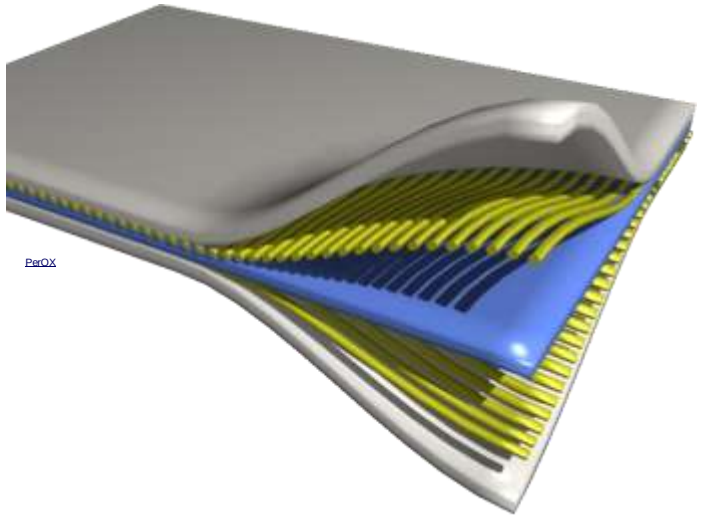
- Manufactured by human being
- No chemical reactions
- Macroscopically heterogeneous
- Advanced material

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# Introduction

## Laminate definition

http://commons.wikimedia.org/wiki/Category:Composite_materials#mediaviewer/File:Composite_3d.png



PerOX

Composite materials composed of several stacked thin layers. These layers are called laminas.

The lamina is the essential unit of composite materials. Typical lamina thickness is around 0.1-1 mm.

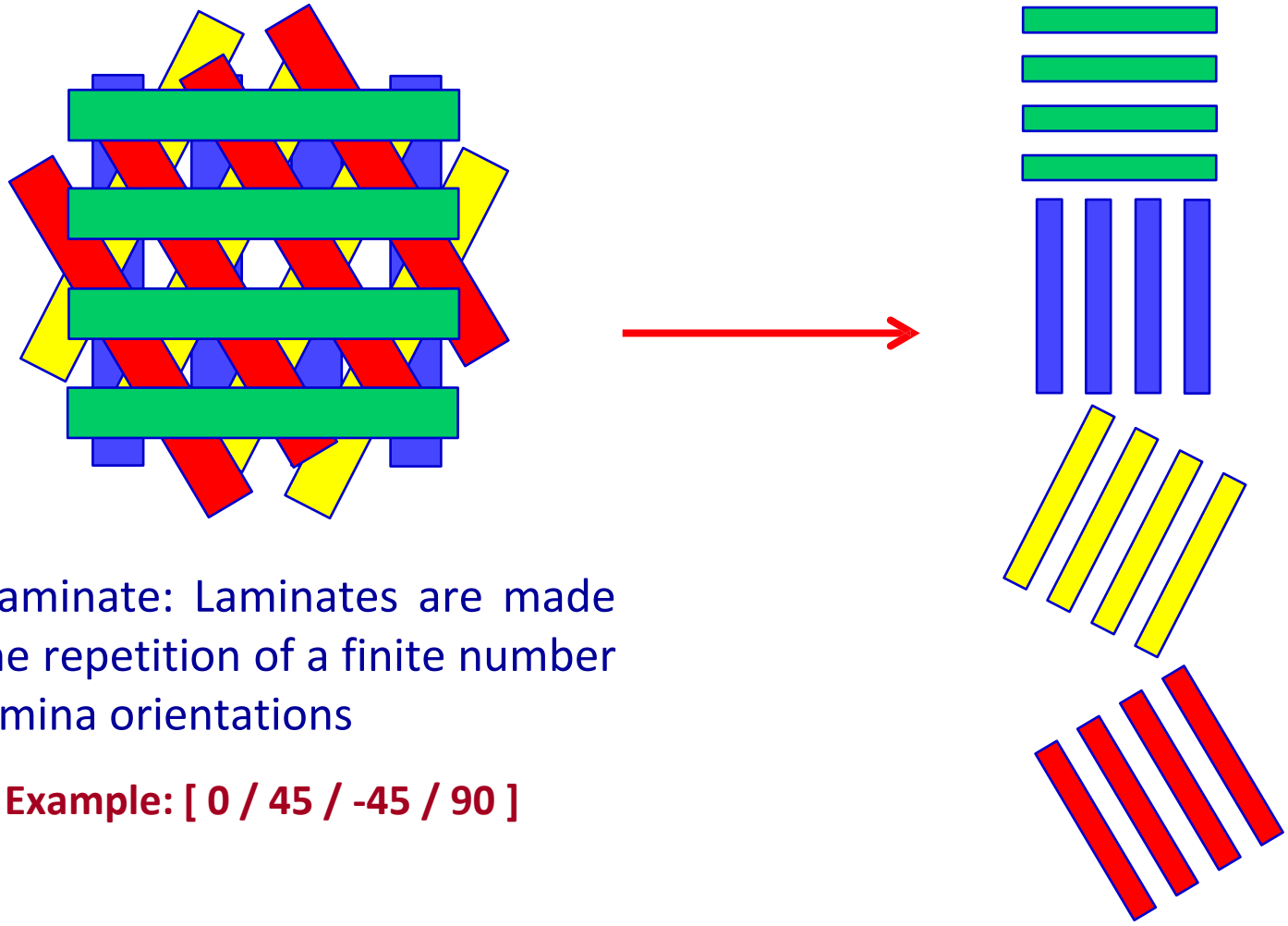
Each lamina can be composed of:

- Short fibres
- Unidirectional Long fibres
- Fabric fibres

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# Introduction

## Laminate definition



Sublaminate: Laminates are made of the repetition of a finite number of lamina orientations

**Example: [ 0 / 45 / -45 / 90 ]**

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# Chapter 4. Laminate and Sandwich Structures

# Laminate theory

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# Anisotropic behaviour

## Linear elastic material

### Duhamel-Neumann formulation

$$\sigma_{ij} = \sum C_{ijkl} \cdot \varepsilon_{kl}$$

Matrix notation:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}$$

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# Anisotropic behaviour

## Linear elastic material

The tensor [C] is symmetric. For anisotropic materials **21** independent elastic constants are required

The inverse tensor can also be applied:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \cdot \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix}$$

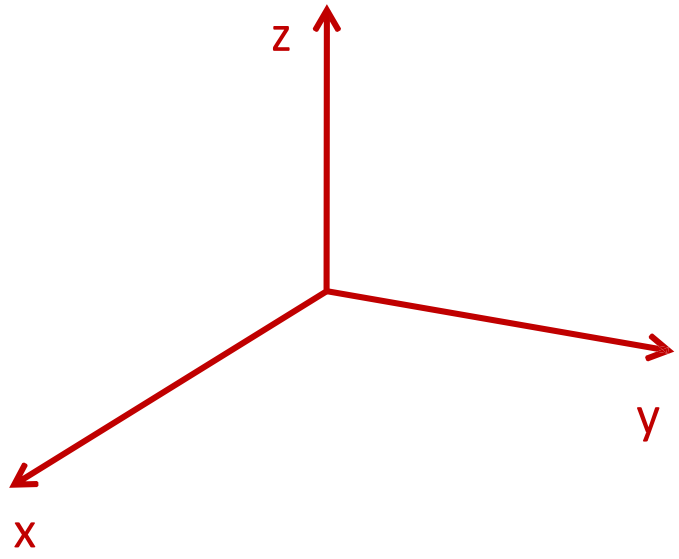
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# Anisotropic behaviour

## Orthotropic material

3 mutually perpendicular planes of elastic symmetry → 9 constants

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}$$




The constants have physical meaning

Planes of elastic symmetry xy xz yz

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# Anisotropic behaviour

## Orthotropic material

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}$$

$$s_{11} = \frac{1}{E_1} \qquad s_{12} = \frac{1}{E_2} \qquad s_{33} = \frac{1}{E_3}$$

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# Anisotropic behaviour

## Orthotropic material

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}$$

$$s_{23} = \frac{1}{G_{23}} \qquad s_{55} = \frac{1}{G_{13}} \qquad s_{66} = \frac{1}{G_{12}}$$

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# Anisotropic behaviour

## Orthotropic material

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}$$

$$s_{12} = -\frac{\nu_{12}}{E_2} \qquad s_{13} = -\frac{\nu_{13}}{E_3} \qquad s_{23} = -\frac{\nu_{23}}{E_3}$$

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# Anisotropic behaviour

## Orthotropic material

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}$$

$$s_{21} = -\frac{\nu_{21}}{E_1} \qquad s_{31} = -\frac{\nu_{31}}{E_1} \qquad s_{32} = -\frac{\nu_{32}}{E_2}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Anisotropic behaviour

## Orthotropic material

**Restrictions to the values of the elastic constants**

**Symmetry of matrix [S]**

$$\frac{\nu_{ji}}{E_i} = \frac{\nu_{ij}}{E_j} \quad , \quad i, j = 1...3 \quad , \quad i \neq j$$

**Terms of main diagonal of [S] and [C] are positives**

$$0 < \nu_{ij} < \sqrt{\frac{E_i}{E_j}} \quad , \quad i, j = 1...3 \quad , i \neq j$$

$$1 - \nu_{12} \cdot \nu_{21} - \nu_{23} \cdot \nu_{32} \quad - \nu_{31} \cdot \nu_{13} - 2 \cdot \nu_{21} \cdot \nu_{32} \cdot \nu_{13} > 0$$

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# Chapter 4. Laminate and Sandwich Structures

## Laminate theory

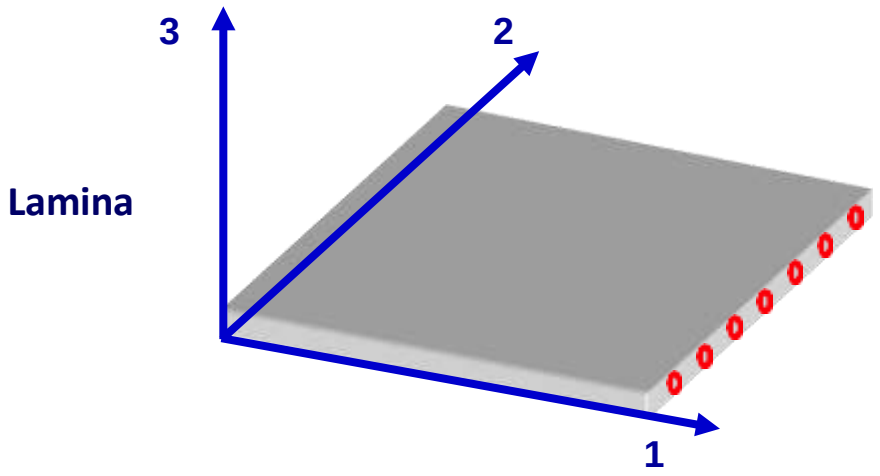
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# Lamina stiffness matrix

## Stiffness matrix in lamina coordinate system (local system)




$$\sigma_{ij} = C_{ijkl} \cdot \varepsilon_{kl}$$

**Plain stress state:**

$$\sigma_3 = 0$$

$$\tau_{13} = 0$$

$$\tau_{23} = 0$$

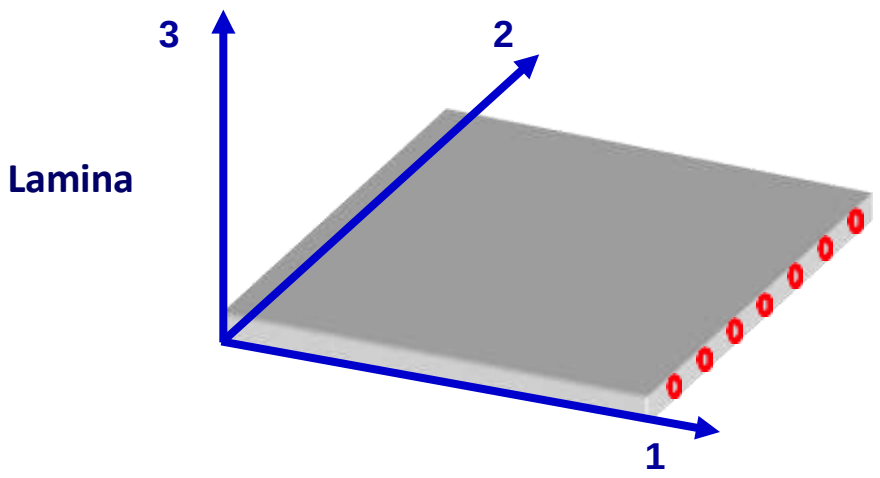
$$[T] = \begin{bmatrix} \sigma_1 & \tau_{12} \\ \tau_{12} & \sigma_2 \end{bmatrix} \qquad \{\sigma\} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \qquad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

¡!

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# Lamina stiffness matrix

## Stiffness matrix in lamina system




$$[Q]=\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{SS} \end{bmatrix}$$

$$[Q]=\begin{bmatrix} \frac{E_1}{1-\nu_{12}\cdot\nu_{21}} & \frac{\nu_{21}\cdot E_2}{1-\nu_{12}\cdot\nu_{21}} & 0 \\ \frac{\nu_{12}\cdot E_1}{1-\nu_{12}\cdot\nu_{21}} & \frac{E_2}{1-\nu_{12}\cdot\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}$$

**Plain stress state:**

$$\{\sigma\}^{12}=[Q]\cdot\{\varepsilon\}^{12}$$

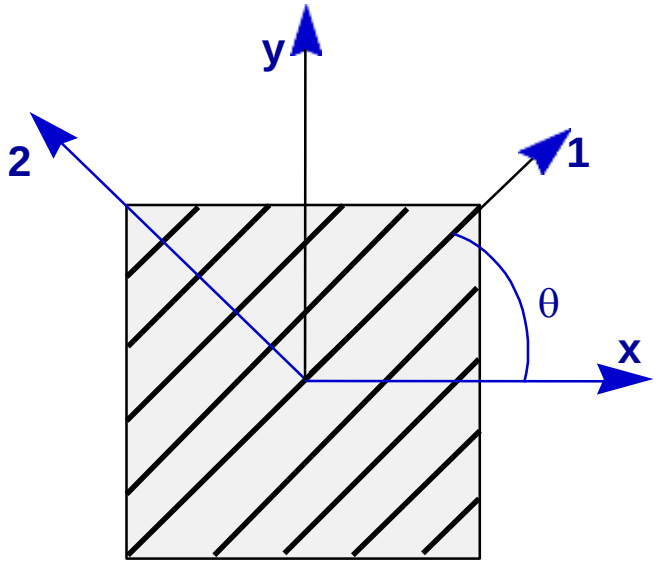
$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}=\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{SS} \end{bmatrix}\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

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# Lamina stiffness matrix

## Stiffness matrix in global system




In axis 12: $\{\sigma\}^{12} = [Q]\cdot\{\varepsilon\}^{12}$

In axis xy: $\{\sigma\}^{xy} = [\overline{Q}]\cdot\{\varepsilon\}^{xy}$

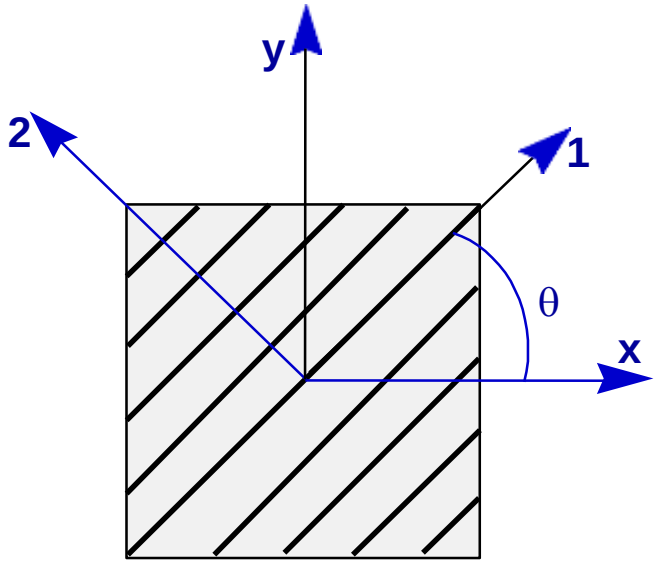
$[Q] \Rightarrow [\overline{Q}]$

¿?

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# Lamina stiffness matrix

## Stiffness matrix in global system




$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [T] \cdot \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\cdot\gamma_{12} \end{Bmatrix} = [T] \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\cdot\gamma_{xy} \end{Bmatrix}$$

$m = \cos\theta \qquad n = \operatorname{sen}\theta$

$$[T] = \begin{bmatrix} m^2 & n^2 & 2\cdot m\cdot n \\ n^2 & m^2 & -2\cdot m\cdot n \\ -m\cdot n & m\cdot n & \left(m^2 - n^2\right) \end{bmatrix}$$

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# Lamina stiffness matrix

## Stiffness matrix in global system

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix}=[Q]\cdot\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=[Q]\cdot\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=[R]^{-1}\cdot\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \frac{1}{2}\cdot\gamma_{xy}\end{Bmatrix}$$

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}=[R]^{-1}\cdot\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \frac{1}{2}\cdot\gamma_{12}\end{Bmatrix}$$

$$[R]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{2}\end{bmatrix}$$

$$[R]^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}$$

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# Lamina stiffness matrix

## Stiffness matrix in global system

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = [T]^{-1}\cdot\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix} = [Q]\cdot\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}$$

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix} = [R]^{-1}\cdot\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \frac{1}{2}\cdot\gamma_{12}\end{Bmatrix}$$

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \frac{1}{2}\cdot\gamma_{12}\end{Bmatrix} = [T]\cdot\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \frac{1}{2}\cdot\gamma_{xy}\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = [T]^{-1}\cdot[Q]\cdot[R]^{-1}\cdot[T]\cdot[R]\cdot\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \begin{bmatrix}\overline{Q}_{xx} & \overline{Q}_{xy} & \overline{Q}_{xs}\\ \overline{Q}_{xy} & \overline{Q}_{yy} & \overline{Q}_{ys}\\ \overline{Q}_{xs} & \overline{Q}_{ys} & \overline{Q}_{ss}\end{bmatrix}\cdot\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

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# Lamina stiffness matrix

## Stiffness matrix in global system

$$\bar{Q}_{xx} = Q_{11} \cdot m^4 + 2 \cdot (Q_{12} + 2 \cdot Q_{SS}) \cdot n^2 \cdot m^2 + Q_{22} \cdot n^4$$

$$\bar{Q}_{yx} = (Q_{11} + Q_{22} - 4 \cdot Q_{SS}) \cdot n^2 \cdot m^2 + Q_{12} \cdot (n^4 + m^4)$$

$$\bar{Q}_{yy} = Q_{11} \cdot n^4 + 2 \cdot (Q_{12} + 2 \cdot Q_{SS}) \cdot n^2 \cdot m^2 + Q_{22} \cdot m^4$$

$$\bar{Q}_{xS} = (Q_{11} - Q_{12} - 2 \cdot Q_{SS}) \cdot n \cdot m^3 + (Q_{12} - Q_{22} + 2 \cdot Q_{SS}) \cdot m \cdot n^3$$

$$\bar{Q}_{yS} = (Q_{11} - Q_{12} - 2 \cdot Q_{SS}) \cdot n^3 \cdot m + (Q_{12} - Q_{22} + 2 \cdot Q_{SS}) \cdot n \cdot m^3$$

$$\bar{Q}_{SS} = (Q_{11} + Q_{22} - 2 \cdot Q_{12} - 2 \cdot Q_{SS}) \cdot n^2 \cdot m^2 + Q_{SS} \cdot (n^4 + m^4)$$

$$m = \cos\theta \qquad n = \operatorname{sen}\theta$$

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# Chapter 4. Laminate and Sandwich Structures

## Laminate theory

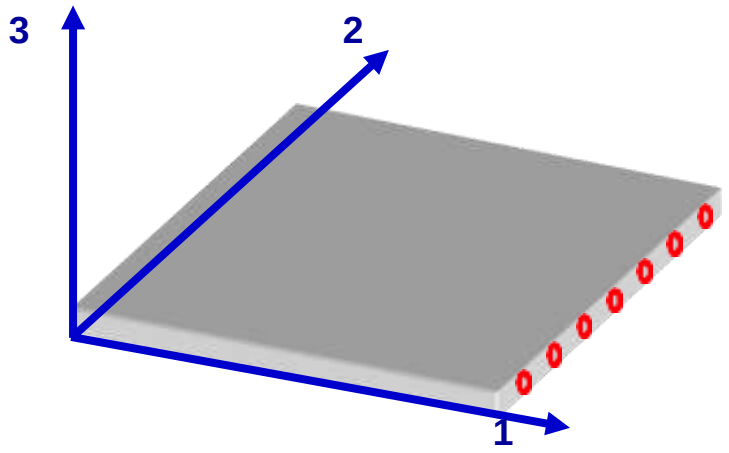
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# Hypotheses

- Each lamina is consider as an orthotropic homogenous material
- The behaviour of lamina material is linear elastic up to failure




$$\{\sigma\}^{12} = [Q]\cdot\{\varepsilon\}^{12}$$

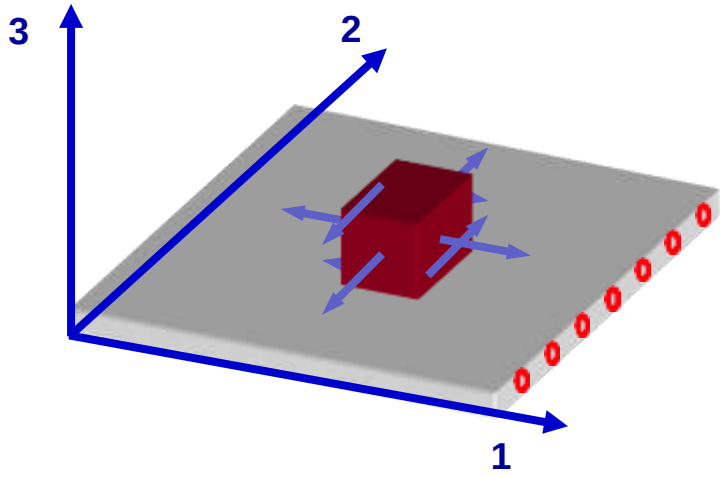
$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{SS} \end{bmatrix}$$

$$[Q] = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\cdot\nu_{21}} & \frac{\nu_{21}\cdot E_2}{1-\nu_{12}\cdot\nu_{21}} & 0 \\ \frac{\nu_{12}\cdot E_1}{1-\nu_{12}\cdot\nu_{21}} & \frac{E_2}{1-\nu_{12}\cdot\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}$$

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# Hypotheses

- **Perfect bonding of laminas**
- **Plain stress state**




$$\sigma_x = 0 \qquad \sigma_x \neq 0$$
$$\tau_{yz} = 0 \qquad \sigma_y \neq 0$$
$$\tau_{xz} = 0 \qquad \tau_{yz} \neq 0$$

This hypothesis is not valid in some situations !!

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# Hypotheses

## Situations where plain stress is not a valid hypothesis

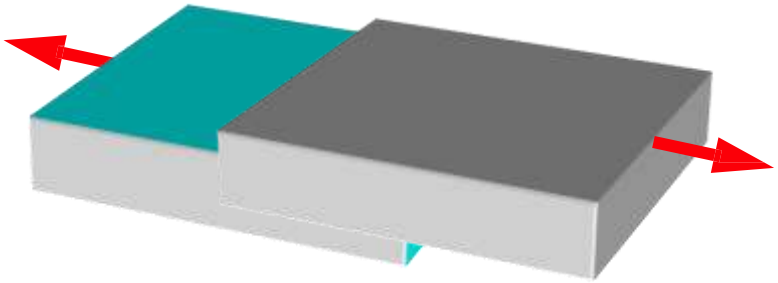
### Free end



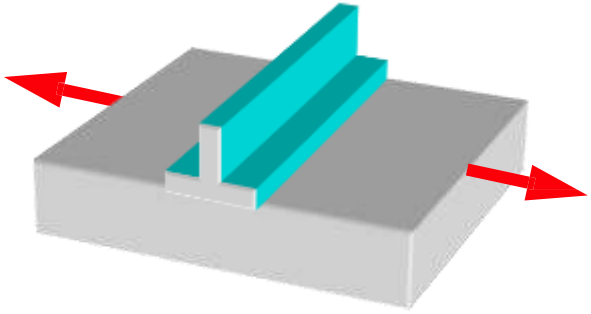
### Variable thickness



### Bonded joints



### Stringers



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# Hypotheses

- **Small displacements**
- **Kirchhoff hypotheses**

A straight line perpendicular to the mid plane remains straight an perpendicular to the mid surface after deformation

$$\gamma_{yz} = 0$$

$$\gamma_{xz} = 0$$

No changes in laminate thickness

$$\varepsilon_z = 0$$

**The displacement of each point can be expressed as a function of the displacements and the rotations of the mid-plane**

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# Chapter 4. Laminate and Sandwich Structures

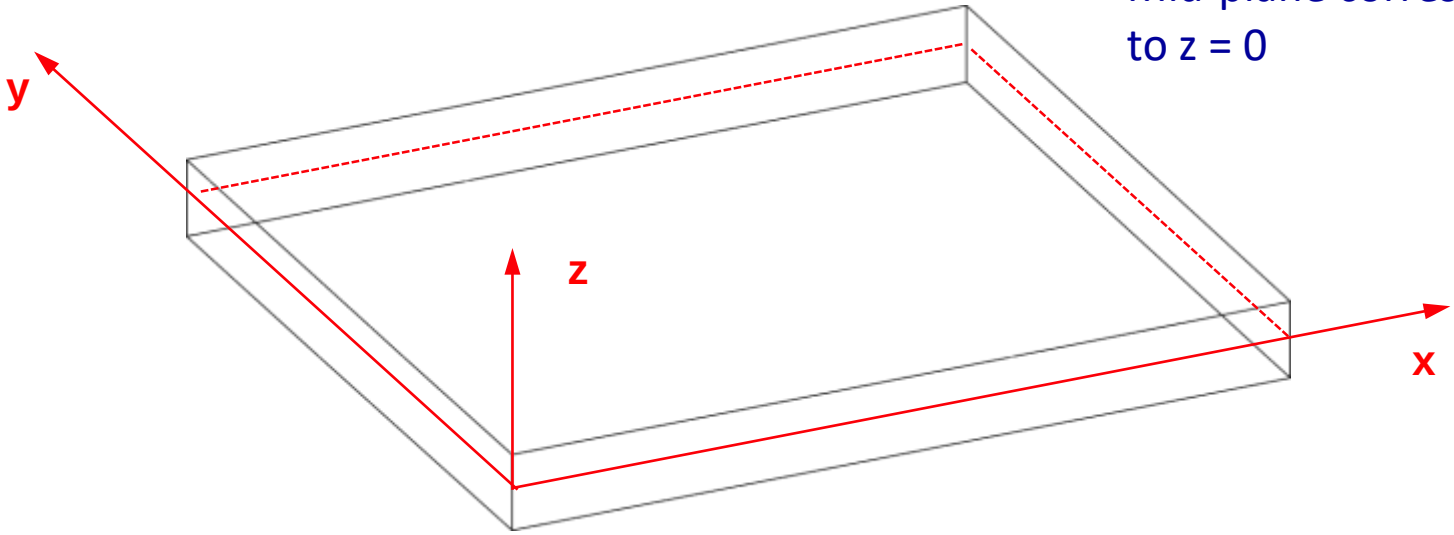
## Laminate theory

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# Laminate kinematics

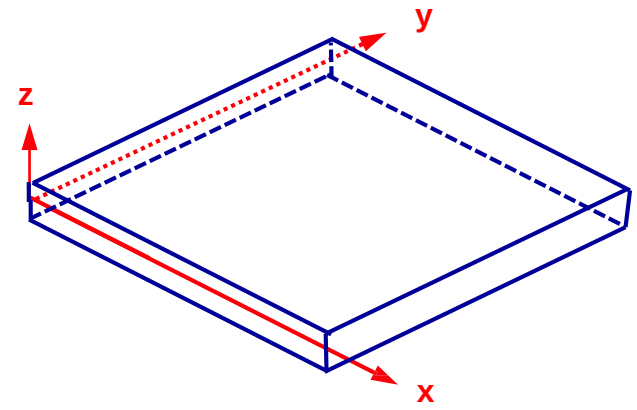
## Definition of laminate axes



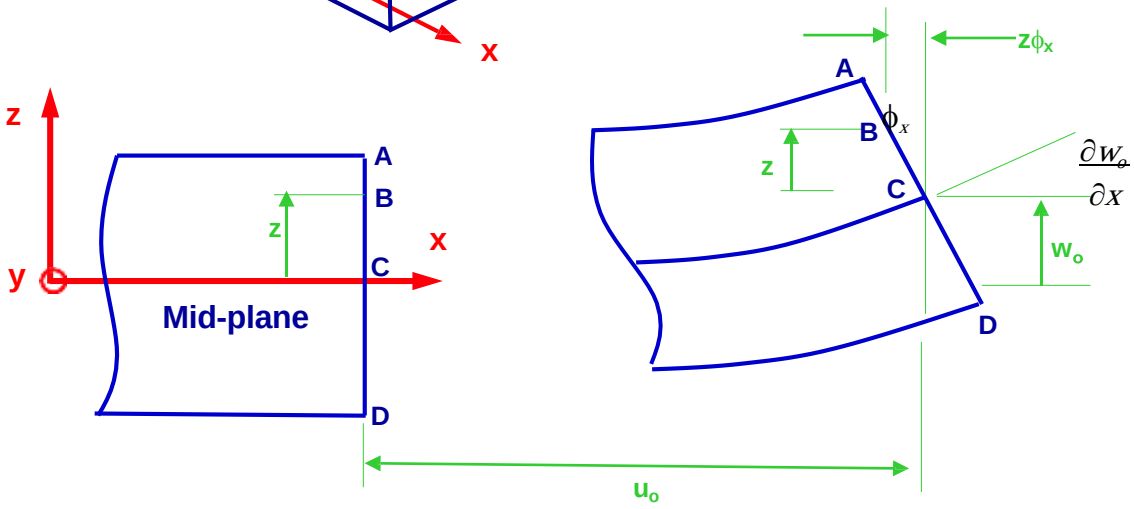

Mid-plane corresponds to z = 0

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# Laminate kinematics




$$\phi_x = \frac{\partial w_o}{\partial x}$$




Undeformed section XZ

Deformed section XZ

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# Laminate kinematics

## Displacements

$$u(x,y,z) = u_o(x,y) - \phi_x \cdot z$$

$$v(x,y,z) = v_o(x,y) - \phi_y \cdot z$$

$$w(x,y,z) = w_o(x,y)$$

**Laminate theory**

$$\phi_x = \frac{\partial w_o}{\partial x}$$

$$\phi_y = \frac{\partial w_o}{\partial y}$$

## Strains

$$\varepsilon_x = \frac{\partial u_o}{\partial x} - z \cdot \frac{\partial^2 w_o}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v_o}{\partial y} - z \cdot \frac{\partial^2 w_o}{\partial y^2}$$

$$\varepsilon_y = 0$$

$$\gamma_{xy} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial y} - z \cdot 2 \cdot \frac{\partial^2 w_o}{\partial x \cdot \partial y}$$

$$\gamma_{xz} = 0$$

$$\gamma_{yz} = 0$$

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# Laminate kinematics

**Strains in mid-plane**

$$\begin{cases} \varepsilon_x^o = \dfrac{\partial u_o}{\partial x} \\ \varepsilon_y^o = \dfrac{\partial v_o}{\partial y} \\ \gamma_{xy}^o = \dfrac{\partial u_o}{\partial y} + \dfrac{\partial v_o}{\partial x} \end{cases}$$

**Curvatures**

$$\begin{cases} kx = -\dfrac{\partial^2 w_o}{\partial x^2} \\ k_y = -\dfrac{\partial^2 w_o}{\partial y^2} \\ k_{xy} = -2\dfrac{\partial^2 w_o}{\partial x \cdot \partial y} \end{cases}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon^o_x \\ \varepsilon^o_y \\ \gamma^o_{xy} \end{Bmatrix} + z \cdot \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix}$$

**Linear variation of strains along thickness direction!!**

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# Chapter 4. Laminate and Sandwich Structures

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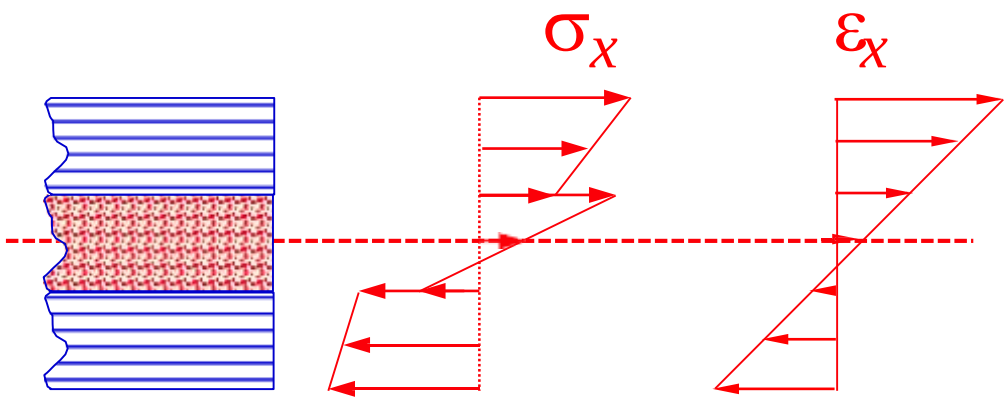
# Internal forces

## Stress distribution

$$\{\sigma\}_i = [\bar{Q}]_i \cdot \{\varepsilon\}$$

$$\{\varepsilon\} = \{\varepsilon^0\} + z \cdot \{k\}$$

**Stress is a discontinuous function of z**

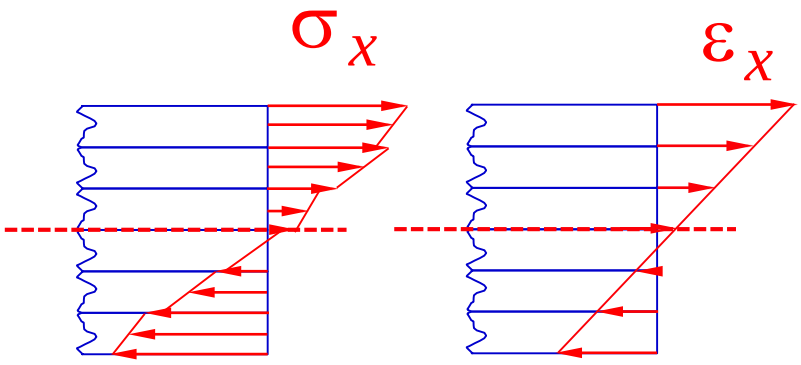



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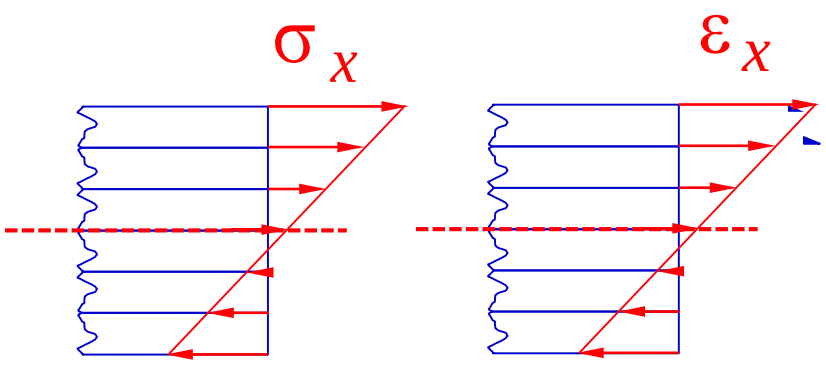
# Internal forces

## Stress distribution

**Laminate**



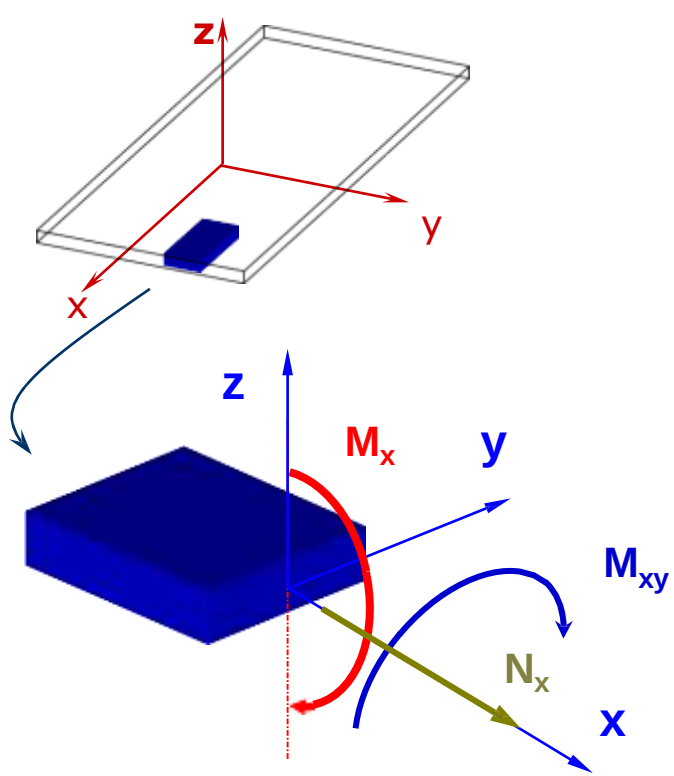

**Isotropic material**

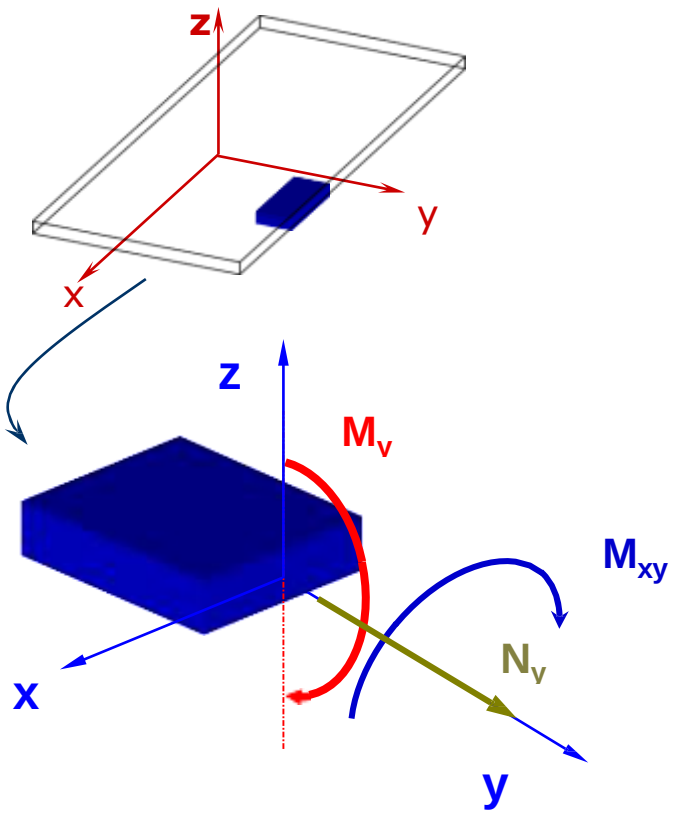



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# Internal forces

## Internal forces defined per unit length

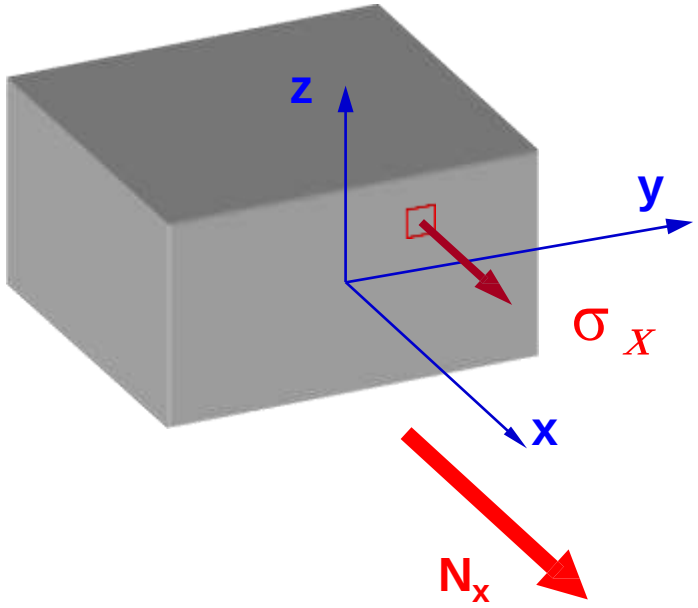



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# Internal forces

## Internal forces defined per unit length




$$N_x = \int_{-h/2}^{h/2} \sigma_x \cdot dz$$

$$N_y = \int_{-h/2}^{h/2} \sigma_y \cdot dz$$

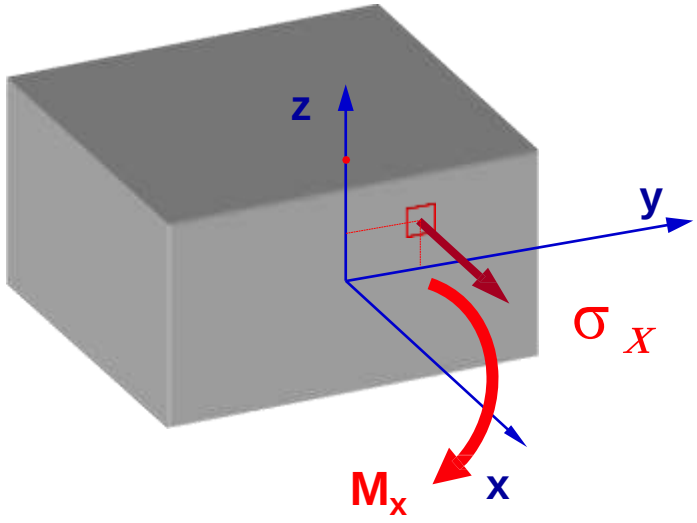
$$\{N\} = \int_{-h/2}^{h/2} \{\sigma\} \cdot dz$$

$$N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} \cdot dz$$

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# Internal forces

## Internal forces defined per unit length




$$M_x = \int_{-h/2}^{h/2} \sigma_x \cdot z \cdot dz$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y \cdot z \cdot dz$$

$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} \cdot z \cdot dz$$

$$\{M\} = \int_{-h/2}^{h/2} \{\sigma\} \cdot z \cdot dz$$

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# Laminate stiffness matrix

$$\{\varepsilon\} = \{\varepsilon^0\} + z \cdot \{k\}$$

$$\{\sigma\}_i = [\bar{Q}]_i \cdot \{\varepsilon\}$$

$$\{N\} = \int_{-h/2}^{h/2} \{\sigma\} \cdot dz$$

$$\{N\} = [A] \cdot \{\varepsilon^0\} + [B] \cdot \{k\}$$

$$[A] = \int_{-h/2}^{h/2} [\bar{Q}]_i \cdot dz$$

$$[B] = \int_{-h/2}^{h/2} [\bar{Q}]_i \cdot z \cdot dz$$

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# Laminate stiffness matrix

$$\{\varepsilon\} = \{\varepsilon^0\} + z \cdot \{k\}$$

$$\{\sigma\}_i = \bar{[Q]}_i \cdot \{\varepsilon\}$$

$$\{M\} = \int_{-h/2}^{h/2} \{\sigma\} \cdot z \cdot dz$$

$$\{M\} = [B] \cdot \{\varepsilon^0\} + [D] \cdot \{k\}$$

$$[D] = \int_{-h/2}^{h/2} \bar{[Q]}_i \cdot z^2 \cdot dz$$

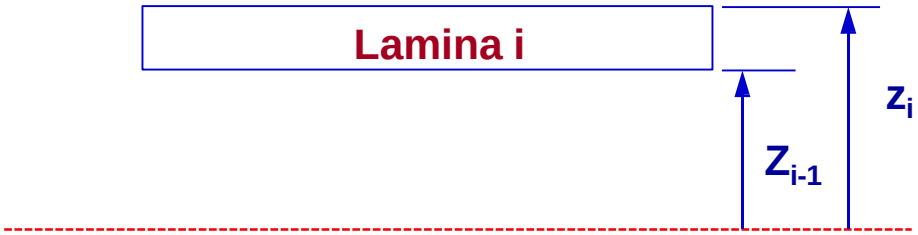
$$[B] = \int_{-h/2}^{h/2} \bar{[Q]}_i \cdot z \cdot dz$$

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# Laminate stiffness matrix

## Plain stiffness matrix [A]

$$[A] = \int_{-h/2}^{h/2} [\bar{Q}]_i \cdot dz \quad \Rightarrow \quad [A] = \sum_{i}^{N} \int_{z_{i-1}}^{z_i} [\bar{Q}]_i \cdot dz$$




$$[A] = \sum_{i}^{N} [\bar{Q}]_i \cdot h_i$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Laminate stiffness matrix

$$[A] = \int_{-h/2}^{h/2} [\bar{Q}]_i \cdot dz$$

$$[D] = \int_{-h/2}^{h/2} [\bar{Q}]_i \cdot z^2 \cdot dz$$

$$[B] = \int_{-h/2}^{h/2} [\bar{Q}]_i \cdot z \cdot dz$$

⇒

Plain stiffness matrix

$$[A] = \sum_i^N [\bar{Q}]_i \cdot h_i$$

Bending stiffness matrix

$$[D] = \frac{1}{3} \cdot \sum_i^N [\bar{Q}]_i \cdot \left( Z_i^3 - Z_{i-1}^3 \right)$$

Coupling stiffness matrix

$$[B] = \frac{1}{2} \cdot \sum_i^N [\bar{Q}]_i \cdot \left( Z_i^2 - Z_{i-1}^2 \right)$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Laminate stiffness matrix

$$\{N\} = [A]\cdot\{\varepsilon^o\} + [B]\cdot\{k\}$$

$$\{\ \} \quad [\ ] \quad \{\ ^o\} \quad [\ ] \quad \{\ \}$$

$$M = B\cdot\varepsilon + D\cdot k$$

$$\{\varepsilon^o\} = [a]\cdot\{N\} + [b]\cdot\{M\}$$

$$\{k\} = [c]\cdot\{N\} + [d]\cdot\{M\}$$

$$[a] = [A]^{-1} - \left\{[B^c]\cdot[D^c]^{-1}\right\}\cdot[C^c]$$

$$[b] = [B^c]\cdot[D^c]^{-1}$$

$$[c] = -[D^c]^{-1}\cdot[C^c]$$

$$[d] = [D^c]^{-1}$$

$$[B^c] = -[A]^{-1}\cdot[B]$$

$$[C^c] = [B]\cdot[A]^{-1}$$

$$[D^c] = [D] - \left\{[B]\cdot[A]^{-1}\right\}\cdot[B]$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Laminate stiffness matrix

## Plain stiffness matrix [A]

$$[A] = \sum_{i}^{N} [\bar{Q}]_i \cdot h_i$$

- Relates axial forces (membrane forces) with plain strains
- Independent on stacking sequence

Laminate $[0/90]_S$ $\Leftrightarrow$ Laminate $[90/0]_S$

Same in plane behaviour

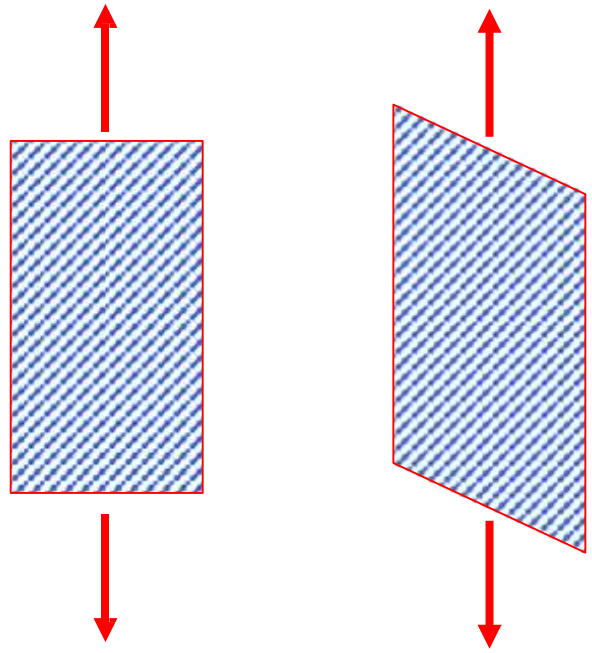
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# Laminate stiffness matrix

**Plain stiffness matrix [A]**

$$[A] = \sum_{i}^{N} [\bar{Q}]_i \cdot h_i$$



- Relates axial forces (membrane forces) with plain strains
- Independent on stacking sequence
- If $A_{1S} \neq 0$ and $A_{2S} \neq 0$ there is a coupling effect between axial and shear forces

$$\begin{Bmatrix} N_x \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{1s} \\ A_{12} & A_{22} & A_{2s} \\ A_{1s} & A_{2s} & A_{ss} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon^o_x \\ \varepsilon^o_y \\ \gamma^o_{xy} \end{Bmatrix}$$

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# Laminate stiffness matrix

## Bending stiffness matrix [D]

$$[D] = \frac{1}{3} \cdot \sum_{i}^{N} [\bar{Q}]_i \cdot \left( z_i^3 - z_{i-1}^3 \right)$$

- Relates bending moments with curvatures
- Depends on stacking sequence

Laminado $[90/0]_S$ $\Leftrightarrow$ Laminado $[0/90]_S$

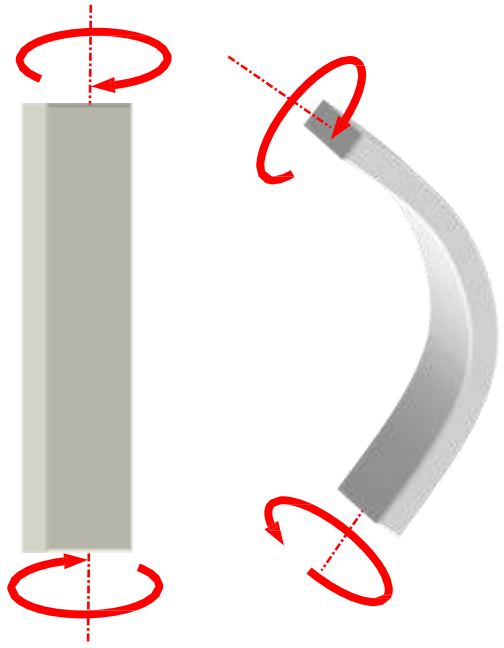
Different bending behaviour

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Laminate stiffness matrix

## Bending stiffness matrix [D]

$$[D] = \frac{1}{3} \cdot \sum_{i}^{N} [\bar{Q}]_i \cdot \left( z_i^3 - z_{i-1}^3 \right)$$



- Relates bending moments with curvatures
- Depends on stacking sequence
- If $D_{1S} \neq 0$ and $D_{2S} \neq 0$ there is a coupling effect between bending and torsion moments

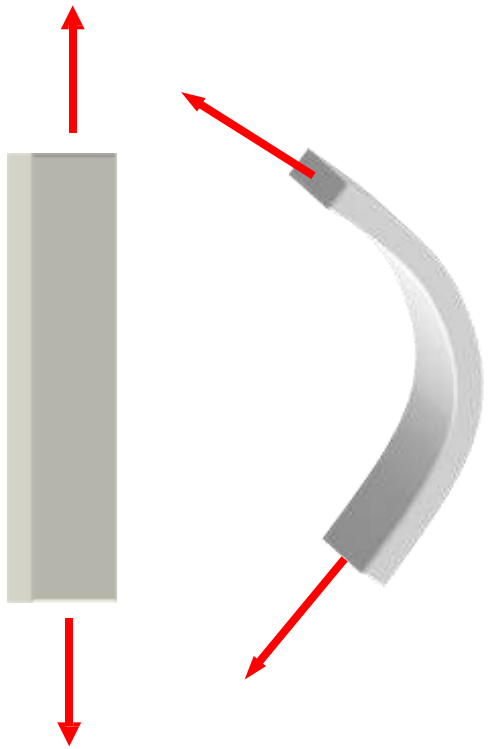
$$\begin{Bmatrix} 0 \\ 0 \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{1s} \\ D & D & D \\ D_{1s}^{12} & D_{2s}^{22} & D_{ss}^{2s} \end{bmatrix} \cdot \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Laminate stiffness matrix

## Coupling stiffness matrix [B]

$$[B] = \frac{1}{2}\sum_i [Q]_i \cdot \left(z_i^2 - z_{i-1}^2\right)$$



- Relates membrane forces with curvatures and moments with plain strains
- Depends on stacking sequence

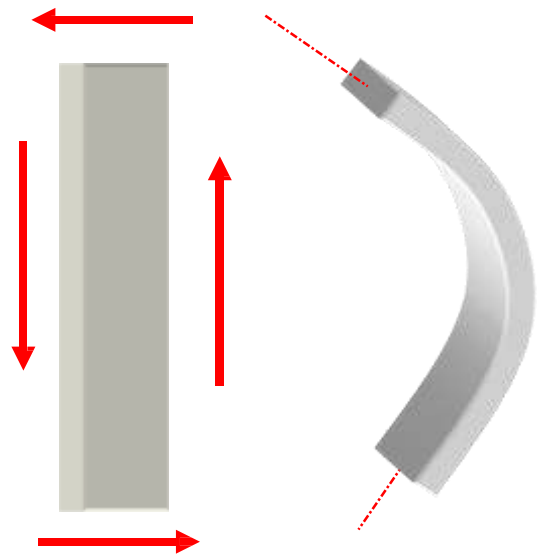
$$\begin{Bmatrix} N_x \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{1s} \\ B_{12} & B_{22} & B_{2s} \\ B_{1s} & B_{2s} & B_{ss} \end{bmatrix} \cdot \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$$

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# Laminate stiffness matrix

## Coupling stiffness matrix [B]

$$[B] = \frac{1}{2}\cdot\sum_{i}^{N}[Q]_i \cdot \left(z_i^2 - z_{i-1}^2\right)$$



- Relates membrane forces with curvatures and moments with plain strains
- Depends on stacking sequence
- If $B_{1S} \neq 0$ and $B_{2S} \neq 0$ there is a coupling effect between axial forces and torsion moment

$$\begin{Bmatrix} 0 \\ 0 \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{1s} \\ B & B & B \\ B_{1s}^{12} & B_{2s}^{22} & B_{ss}^{2s} \end{bmatrix} \cdot \begin{Bmatrix} K_x \\ K^y \\ K_{xy} \end{Bmatrix}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Chapter 4. Laminate and Sandwich Structures

## Laminate theory

1. Introduction
2. Anisotropic behaviour
3. Lamina stiffness matrix
4. Hypotheses
5. Laminate kinematics
6. Internal forces
7. Laminate stiffness matrix
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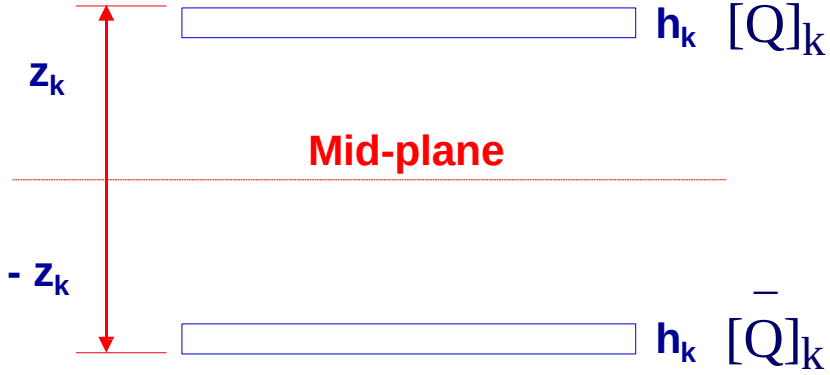
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# Laminate configurations

## Symmetric laminate

The stacking sequence is symmetrical with respect to the laminate mid-plane




$$[B] = \frac{1}{2}\cdot\sum_{i}^{N}\bar{[Q]}_i \cdot \left( Z_i^2 - Z_{i-1}^2 \right)$$

$$[B] = [0]$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Laminate configurations

## Symmetric laminate

The stacking sequence is symmetrical with respect to the laminate mid-plane




$$\{N\} = [A] \cdot \{\varepsilon^0\}$$
$$\{M\} = [D] \cdot \{k\}$$

$$[a] = [A]^{-1}$$
$$[d] = [D]^{-1}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Laminate configurations

## Equilibrated laminate

Same number of laminas oriented at $+\theta$ and a $-\theta$

An equilibrated laminate can be symmetric, asymmetric or antisymmetric

Symmetric: $[\pm\theta_1/\ \pm\theta_2]_S$ (8 plies)

Antisymmetric: $[\theta_1/\ \theta_2/\ -\theta_1/\ -\theta_2]$ (4 plies)

Asymmetric: $[\theta_1/\ -\theta_2/\ -\theta_1/\ \theta_2]$ (4 plies)

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# Laminate configurations

## Orthotropic laminate

Orthotropic lamina axes correspond to laminate axis

In orthotropic laminates: $A_{xs} = A_{ys} = 0$

In orthotropic symmetric laminates: $D_{xs} = D_{ys} = 0$

Example:

$[0/90/90/0]_S$

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# Laminate configurations

## Quasi-isotropic laminate

A laminate is quasi-isotropic if:

$$A_{xx} = A_{yy}$$

$$A_{xx} - A_{xy} = 2 \cdot A_{ss}$$

$$A_{xs} = A_{ys} = 0$$

Example:

$[0/90/+45/-45]_S$

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# Laminate equivalent constants

## Chapter 4. Laminate and Sandwich Structures

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# Laminate equivalent constants

## Membrane forces (symmetric laminates)

Average stresses in laminate

$$\sigma_x^o = \frac{N_x}{H}$$

$$\sigma_y^o = \frac{N_y}{H}$$

$$\tau_{xy}^o = \frac{N_{xy}}{H}$$

$$H = \sum_i^N h_i$$

$$\{N\} = [A] \cdot \{\varepsilon^o$$

$$\{\sigma^o\} = [A^*] \cdot \{\varepsilon^o$$

Normalized plain stiffness matrix

$$[A^*] = \frac{[A]}{H}$$

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# Laminate equivalent constants

## Membrane forces (symmetric laminates)

$$\{\sigma^0\} = [A^*] \cdot \{\varepsilon^0\}$$

$$\{\varepsilon^0\} = [a^*] \cdot \{\sigma^0\}$$

$$\begin{Bmatrix} \varepsilon^o_x \\ \varepsilon^o_y \\ \gamma^o_{xy} \end{Bmatrix} = \begin{bmatrix} a^*_{11} & a^*_{12} & a^*_{1s} \\ a^*_{21} & a^*_{22} & a^*_{2s} \\ a^*_{s1} & a^*_{s2} & a^*_{ss} \end{bmatrix} \cdot \begin{Bmatrix} \sigma^o_x \\ \sigma^o_y \\ \tau^o_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{12}/E_2 & 0 \\ -\nu_{21}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

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# Laminate equivalent constants

## Membrane forces (symmetric laminates)

### Engineering plain constants

$$E_1^o = \frac{1}{a^*_{11}} \qquad \nu_{21}^o = -\frac{a^*_{21}}{a^*_{11}}$$

$$E_2^o = \frac{1}{a^*_{22}} \qquad \nu_{12}^o = -\frac{a^*_{12}}{a^*_{22}}$$

$$G_{12}^o = \frac{1}{a^*_{ss}}$$

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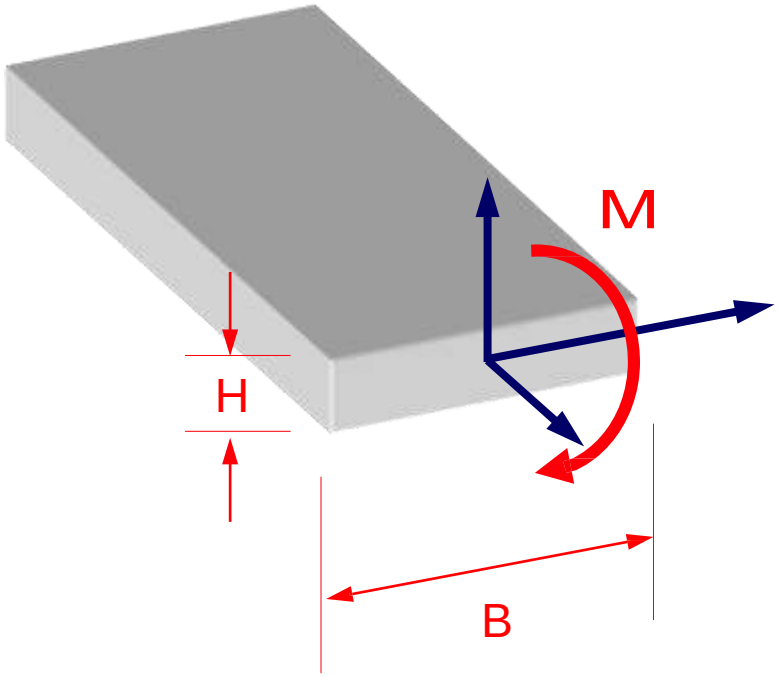
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# Laminate equivalent constants

## Bending and torsion moments (symmetric laminates)

Isotropic material:

$$\sigma_x = \frac{M \cdot z}{I} \qquad I = \frac{1}{12} \cdot B \cdot H^3$$




$$\varepsilon_x\Big|_{max.} = \frac{H}{2} \cdot k_x$$

$$\sigma_x\Big|_{max.} = \frac{6}{B \cdot H^2} \cdot M$$

$$M_x = \frac{M}{B}$$

$$\sigma_x\Big|_{max.} = \frac{6}{H^2} \cdot M_x$$

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# Laminate equivalent constants

## Bending and torsion moments (symmetric laminates)

Maximum (average) stresses in laminate

$$\sigma_x^f = \frac{6 \cdot M_x}{H^2}$$

$$\sigma_y^f = \frac{6 \cdot M_y}{H^2}$$

$$\tau_{xy}^f = \frac{6 \cdot M_{xy}}{H^2}$$

Maximum strains in laminate

$$\{\varepsilon\}_{\max.} = z_{\max.} \cdot \{k\}$$

$$\varepsilon_x^f = \frac{H \cdot k_x}{2}$$

$$\varepsilon_y^f = \frac{H \cdot k_y}{2}$$

$$\gamma_{xy}^f = \frac{H \cdot k_{xy}}{2}$$

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# Laminate equivalent constants

## Bending and torsion moments (symmetric laminates)

$$\{M\} = [D] \cdot \{k\}$$

$$\{\sigma^f\} = [D^*] \cdot \{\varepsilon^f$$

Normalized bending stiffness matrix $$[D^*] = \frac{12 \cdot [D]}{H^3}$$

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# Laminate equivalent constants

## Bending and torsion moments (symmetric laminates)

$$\left\{\sigma^f\right\} = \left[D^*\right] \cdot \left\{\varepsilon^f\right.$$

$$\Downarrow$$

$$\left\{\varepsilon^f\right\} = \left[d^*\right] \cdot \left\{\sigma^0\right.$$

$$\begin{Bmatrix} \varepsilon^f_x \\ \varepsilon^f_y \\ \gamma^f_{xy} \end{Bmatrix} = \begin{bmatrix} d^*_{11} & d^*_{12} & d^*_{1s} \\ d^*_{21} & d^*_{22} & d^*_{2s} \\ d^*_{s1} & d^*_{s2} & d^*_{ss} \end{bmatrix} \cdot \begin{Bmatrix} \sigma^f_x \\ \sigma^f_y \\ \tau^f_{xy} \end{Bmatrix}$$

$$\Updownarrow$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{12}/E_2 & 0 \\ -\nu_{21}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

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# Laminate equivalent constants

## Bending and torsion moments (symmetric laminates)

### Engineering bending constants

$$E_1^f = \frac{1}{d_{11}^*} \qquad \nu_{21}^f = -\frac{d_{21}^*}{d_{11}^*}$$

$$E_2^f = \frac{1}{d_{22}^*} \qquad \nu_{12}^f = -\frac{d_{12}^*}{d_{22}^*}$$

$$G_{12}^f = \frac{1}{d_{ss}^*}$$

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# Laminate equivalent constants

## Asymmetric laminate

$$\{N\} = [A]\cdot\{\varepsilon^0\} + [B]\cdot\{k\}$$

$$\{M\} = [B]\cdot\{\varepsilon^0\} + [D]\cdot\{k\}$$

$$\{\sigma^0\} = \frac{\{N\}}{H} \qquad \{\varepsilon^f\} = \frac{H\cdot\{k\}}{2}$$

$$\{\sigma^f\} = \frac{6\cdot\{M\}}{H^2}$$

$$\{\sigma^0\} = [A^*]\cdot\{\varepsilon^0\} + [B^*]\cdot\{\varepsilon^f\}$$

$$\{\sigma^f\} = 3\cdot[B^*]\cdot\{\varepsilon^0\} + [D^*]\cdot\{\varepsilon^f\}$$

$$[A^*] = \frac{[A]}{H} \qquad [D^*] = \frac{12\cdot[D]}{H^3}$$

Normalized coupling stiffness matrix

$$[B^*] = \frac{2\cdot[B]}{H^2}$$

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# References

**Stress analysis of fiber-reinforced composite materials**
M. W. Hyler
McGraw Hill, 1998
Cap.4 The plane-stress assumption
Cap.6 Classical Lamination theory: The kirchhoff Hypothesis

**Engineering Mechanics of Composite Materials**
I.M. Daniel y O. Ishai,
Oxford University Press
Cap.5 Elastic behaviour of multidirectional laminates

**Mechanics of laminated composite plates. Theory and analysis**
J.N. Reddy
CRC Press. 1997
Cap 5: Classical and First-order theories of laminated composite plates

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# Aerospace Structures

## Chapter 4. Laminate and sandwich structures

### Composite Beams

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## Chapter 4. Composite beams and plates

## Composite beams

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3. Stresses in solid beams
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# Chapter 4. Composite beams and plates

# Composite beams

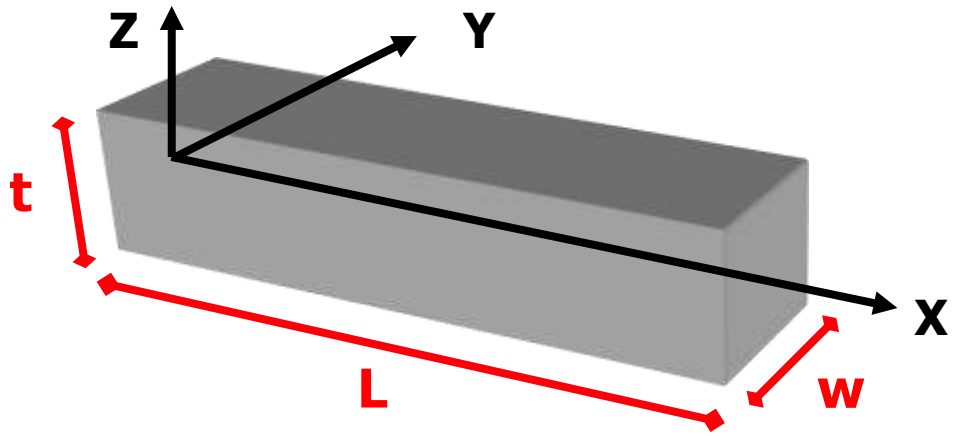
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# Introduction

## Previous knowledge




L = Length

t = Thickness

w = Width

- w / L << 1
- t / L << 1
- Loads directions are in plane x-z.
- If loads directions are in plane y-z, we can apply analogous formulation.
- Principle of superposition can be use for loads direction in plane x-z and plane y-z.

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# Introduction

Internal force in beams:

- Tensile
- Compressive
- Bending moment
- Torsion moment
- Shear
- Combined forces:
  Bending-shear, bending-torsion, …

**Manufacture methods**

- Pultrusion
- Rolling
- Prepegç
- Resin injection

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# Introduction

- Design:
  - Minimize cross-section (weight)
  - Maximize moments of inertia
    - Increase section dimensions
    - Decrease thickness
  - Beams modelled as prismatic elements

- Design stages:

  - Beam dimensions, under quasi-static or dynamic

  - Joint of beam with the rest of the structure

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# Chapter 4. Composite beams and plates

# Composite beams

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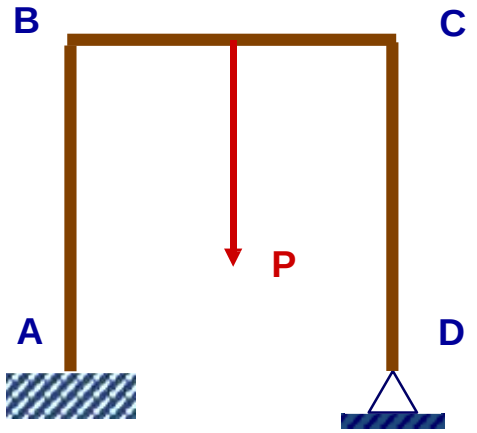
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# Analysis of beams

## Calculation process

– Classical beam theory can be used to analyse the structure

- Degree of static indeterminacy
- Reactions calculation
- Internal forces laws





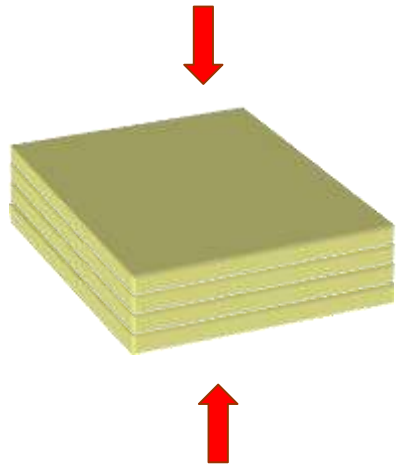

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# Analysis of beams

## Calculation process

– Stresses analysis:

- Laminate theory or first-order-shear-deformation theory can be used to calculate beam strains
- Strains and stresses can be found for each layer
- A failure criterion is applied




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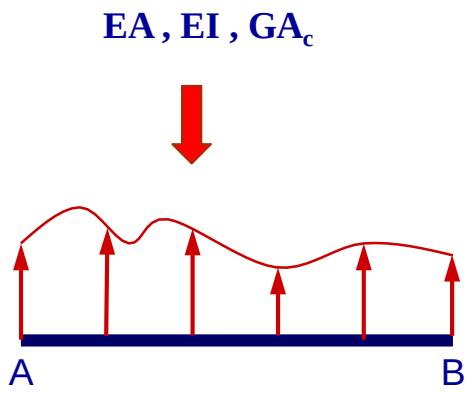
# Analysis of beams

## Calculation process

– Displacement analysis

- Laminate equivalent elastic properties are found using laminate theory or first-order-shear-deformation theory to calculate stiffness matrices

- Navier-Bresse formulation, Mohr theorems or energetic methods can be used to found the displacements




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# Chapter 4. Composite beams and plates

# Composite beams

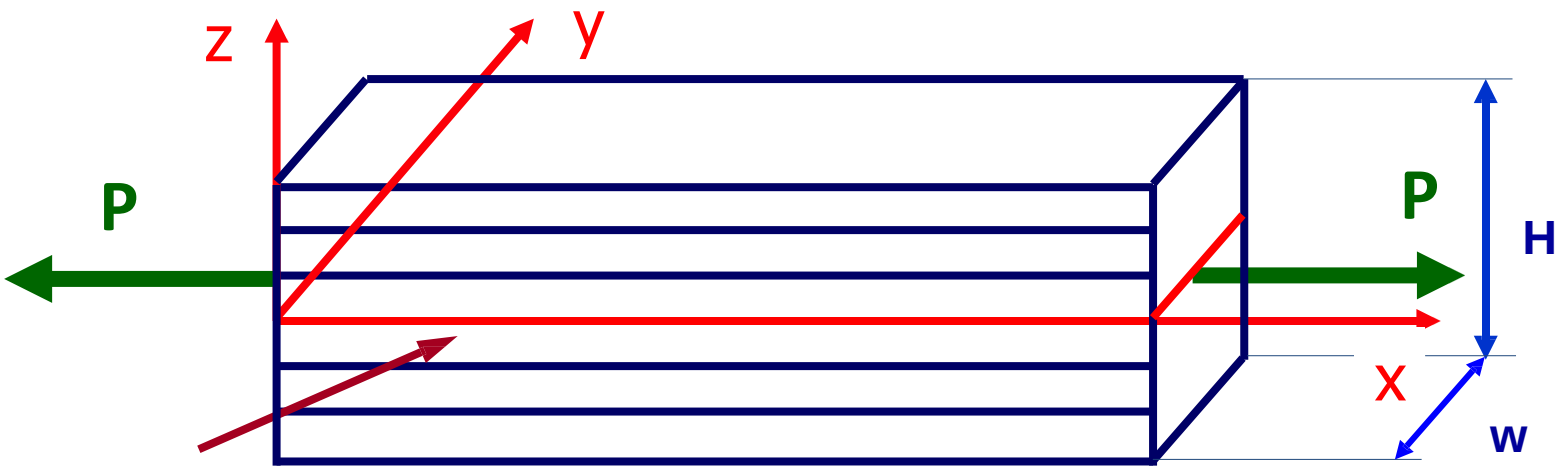
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7. References

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# Stresses in solid beams

## Beams under tensile/compressive loads




**P**: External load (N)

N: Internal axial force per unit length (N/m)

$$N_x = N = \frac{P}{w}$$

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# Stresses in solid beams

## Beams under tensile/compressive loads

Laminate theory:

$$\begin{Bmatrix} \\ N \end{Bmatrix} = \underbrace{\begin{bmatrix} \int_{-t/2}^{t/2} [\overline{Q}] \cdot dz \end{bmatrix}}_{[A]} \cdot \begin{Bmatrix} \\ \varepsilon^0 \end{Bmatrix}$$

$$\{N\} = [A] \cdot \{\varepsilon^0\} \; in \quad \text{N/m}$$

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# Stresses in solid beams

## Beams under tensile/compressive loads

Load state:

$$\begin{Bmatrix} N \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{1S} \\ A_{12} & A_{22} & A_{2S} \\ A_{1S} & A_{2S} & A_{SS} \end{bmatrix} \begin{Bmatrix} \varepsilon^0_x \\ \varepsilon^0_y \\ \gamma^0_{xy} \end{Bmatrix}$$

Strains:

$$\begin{Bmatrix} \varepsilon^0_x \\ \varepsilon^0_y \\ \gamma^0_{xy} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{1S} \\ a_{12} & a_{22} & a_{2S} \\ a_{1S} & a_{2S} & a_{SS} \end{bmatrix} \begin{Bmatrix} N \\ 0 \\ 0 \end{Bmatrix}$$

$$\varepsilon^0_x = a_{11} \cdot N$$

$$N = N(x)$$

$$\varepsilon^0_y = 0 \qquad \gamma^0_{xy} = 0$$

$$\varepsilon_x = \varepsilon_x(x)$$

One-dimensional problem

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Stresses in solid beams

## Beams under tensile/compressive loads

Strains of each lamina

Lamina k-th:

$$\{\varepsilon\}^k = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^k = \begin{Bmatrix} \varepsilon^0_x \\ \varepsilon^0_y \\ \gamma^0_{xy} \end{Bmatrix} = \{\varepsilon^0\}$$

Stresses in each lamina

$$\{\sigma\}^k = [\overline{Q}]_k \{\varepsilon\}^k = [\overline{Q}]_k \{\varepsilon^0\}$$

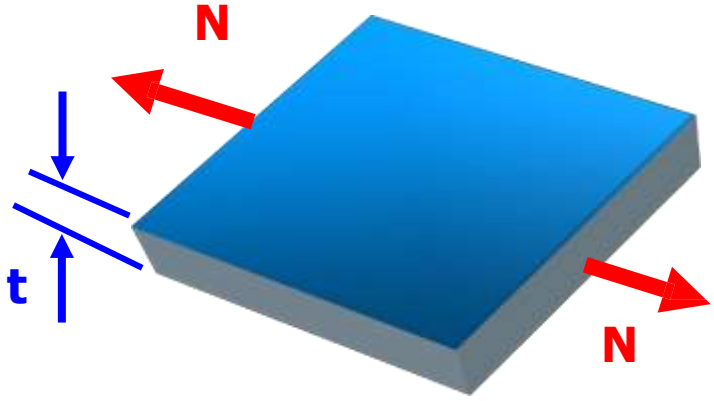
**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Stresses in solid beams

## Beams under tensile/compressive loads

### Laminate equivalent constants

$$\varepsilon_x^0 = a_{11} \cdot N \qquad \sigma_x^0 = \frac{N}{t} = \frac{\mathbf{P}}{t \cdot w}$$




### In isotropic materials

$$\sigma_x = E \cdot \varepsilon_x \qquad \sigma_x = \frac{\mathbf{P}}{A}$$

$$\Downarrow$$

$$E_x^0 = \frac{1}{a_{11} \cdot t} = \frac{1}{a_{11}^*} \qquad A = w \cdot t$$

$$\left[a^*\right] = \frac{1}{\left[A^*\right]} \qquad \left[A^*\right] = \frac{[A]}{t}$$

$$\langle EA \rangle = \frac{w}{a_{11}}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Stresses in solid beams

## Beams under tensile/compressive loads

### Approximate solution

$$\{\varepsilon\}^k = \{\varepsilon^0\} = [S]^{-1}\{\sigma\}^k$$

$$\sigma_x^k = E_x^k \, \varepsilon_x^0$$

$$\varepsilon_x^0 = \frac{\sigma_x^k}{E_x^k}$$

$$\sigma_x^k \cdot t_k = E_x^k \cdot t_k \cdot \varepsilon_x^0$$

$$N = \sum_k \sigma_x^k \cdot t_k = \left(\sum_k E_x^k \cdot t_k\right) \cdot \varepsilon_x^0$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Stresses in solid beams

## Beams under tensile/compressive loads

Where:

$$E_{\mathrm{x}}^{0} = \frac{\sum_{k} E_{\mathrm{x}}^{k} \cdot t_{k}}{t}$$

$$N = \sum_{k} \sigma_{x}^{k} \cdot t_{k} = \left(\sum_{k} E_{\mathrm{x}}^{k} \cdot t_{k}\right) \cdot \varepsilon_{\mathrm{x}}^{0} = E_{\mathrm{x}}^{0} \cdot t \cdot \varepsilon_{\mathrm{x}}^{0}$$

$$\sigma_{x}^{0} = \frac{N}{t} = E_{\mathrm{x}}^{0} \cdot \varepsilon_{\mathrm{x}}^{0}$$

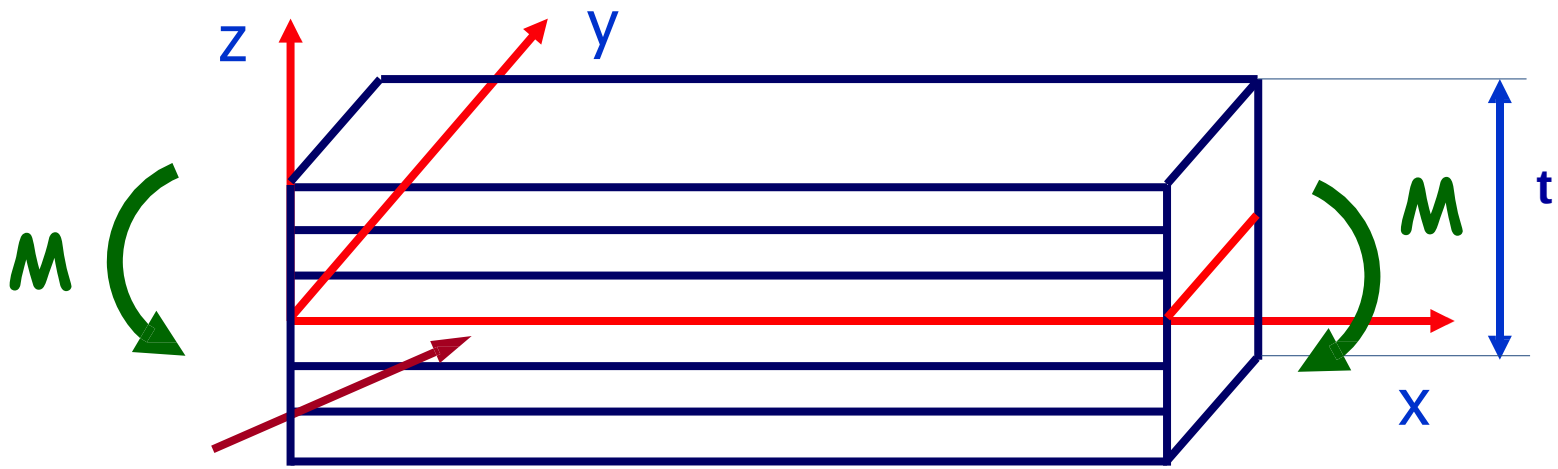
$$\varepsilon_{\mathrm{x}}^{0} = \frac{1}{E_{\mathrm{x}}^{0}} \frac{N}{t}$$

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# Stresses in solid beams

## Bending of beams




**M**: External moment (N·m)

$M_x$: Bending moment per unit length (N)

$$M_x = M = \frac{\mathbf{M}}{w}$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Stresses in solid beams

## Bending of beams

**(Narrow beams)**

Load state:

$$\{M\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{Bmatrix} M \\ 0 \\ 0 \end{Bmatrix}$$

Laminate theory:

Symmetric laminate under pure bending moment

$$\begin{Bmatrix} M \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{1S} \\ D_{12} & D_{22} & D_{2S} \\ D_{1S} & D_{2S} & D_{SS} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Stresses in solid beams

## Bending of beams

**Laminate curvatures:**

$$\begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{1S} \\ d_{12} & d_{22} & d_{2S} \\ d_{1S} & d_{2S} & d_{SS} \end{bmatrix} \begin{Bmatrix} M \\ 0 \\ 0 \end{Bmatrix}$$

$$K_x = d_{11} \cdot M$$

$$M = M(x)$$

$$k_x = k_x(x)$$

**One-dimensional problem**

**Authors:** Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Stresses in solid beams

## Bending of beams

### Strains in each lamina

Lamina k-th:

$$\{\varepsilon\}^k = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^k = z \cdot \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = z \cdot \{k\}$$

### Stresses in each lamina

$$\{\sigma\}^k = [\bar{Q}]_k \{\varepsilon\}^k$$

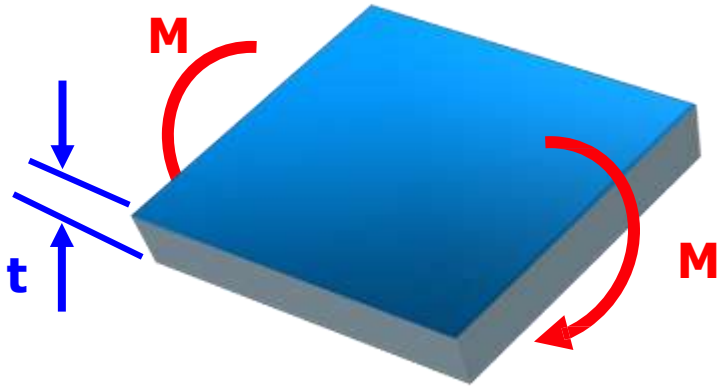
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

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# Stresses in solid beams

## Bending of beams

### Laminate equivalent constants




$$\kappa_x = d_{11} \cdot M = \frac{1}{\rho_x}$$

In isotropic materials:

$$\frac{1}{\rho_x} = \frac{M}{E \cdot I}$$

$$I_y = \frac{1}{12} \cdot w \cdot t^3$$

$$E_x^b = \frac{12}{t^3 \cdot d_{11}} = \frac{1}{d_{11}^*}$$

$$[d^*] = \frac{1}{[D^*]} \qquad [D^*] = \frac{12 \cdot D}{t^3}$$

$$\langle EI \rangle = \frac{w}{d_{11}}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Stresses in solid beams

## Bending of beams

### Laminate average maximum stresses

In symmetrical laminates under bending an equivalent maximum stresses can be defined as:

$$\sigma_x^b = \frac{6M}{t^2} \qquad \sigma_y^b = 0 \qquad \tau_x^b = 0$$

$$\{\sigma\}^b = \begin{Bmatrix} \frac{6M}{t^2} \\ 0 \\ 0 \end{Bmatrix} = \frac{6}{t^2}\{M\}$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Stresses in solid beams

## Bending of beams

### Laminate maximum strains

Equivalent maximum strains:

$$\{\varepsilon\}^b = \frac{t}{2}\cdot\{\kappa\}$$

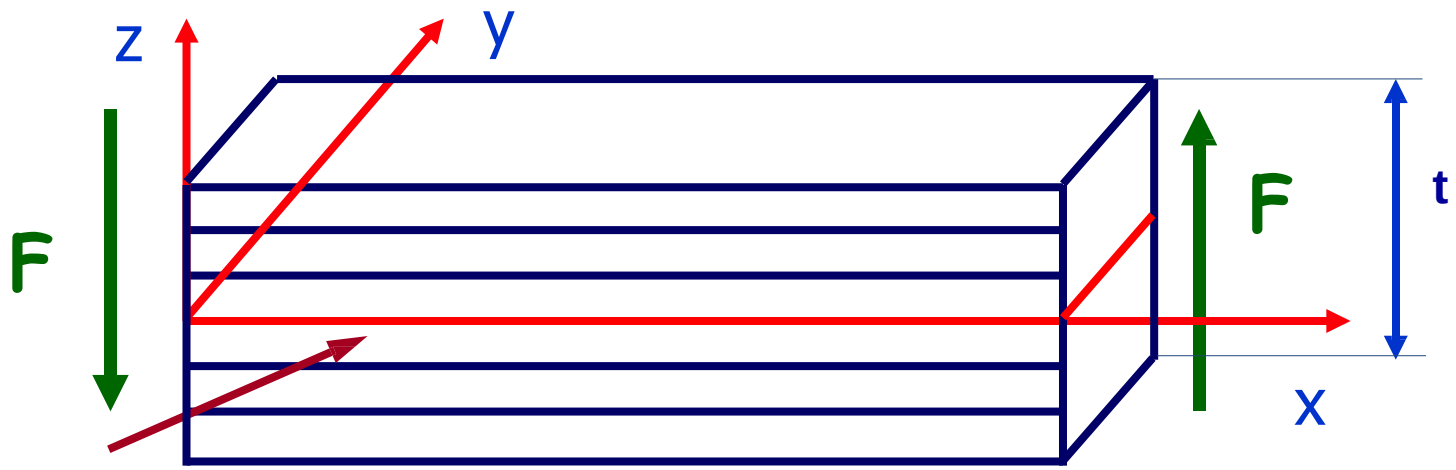
$$\{\varepsilon\}^b = \frac{t}{2}\cdot\{\kappa\} = \frac{t}{2}[D]^{-1}\{M\} \quad \Rightarrow \quad \begin{cases} \varepsilon_x^b = \frac{t}{2} d_{11} M \\ \varepsilon_y^b = \frac{t}{2} d_{12} M \\ \gamma_{xy}^b = \frac{t}{2} d_{1S} M \end{cases}$$

Since a linear through-the-thickness strains variation is assumed, strains at any layer can be found. Therefore stresses can be also found.

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Stresses in solid beams

## Interlaminar shear forces in beams




**F**: External force (N)

$N_{xy}$: Shear force per unit length (N/m)

$$Q_x = Q = \frac{F}{w}$$

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# Stresses in solid beams

## Interlaminar shear forces in beams

Load state:

$$\{Q\} = \begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{Bmatrix} 0 \\ Q \end{Bmatrix}$$

First-Order Shear Deformation theory:

$$\begin{Bmatrix} 0 \\ Q \end{Bmatrix} = K \begin{bmatrix} A_{44} & A_{45} \\ A_{54} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

Strains:

$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \frac{1}{K} \begin{bmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{bmatrix} \begin{Bmatrix} 0 \\ Q \end{Bmatrix} \qquad \gamma_{xz} = \frac{a_{55}}{K} \cdot Q$$

**One-Dimensional problem**

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

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# Stresses in solid beams

## Interlaminar shear forces in beams

Lamina strains

**Lamina k-th:**

$$\{\gamma\}^k = \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^k = \begin{Bmatrix} \gamma^0_{yz} \\ \gamma^0_{xz} \end{Bmatrix} = \{\gamma^0\}$$

Lamina stress:

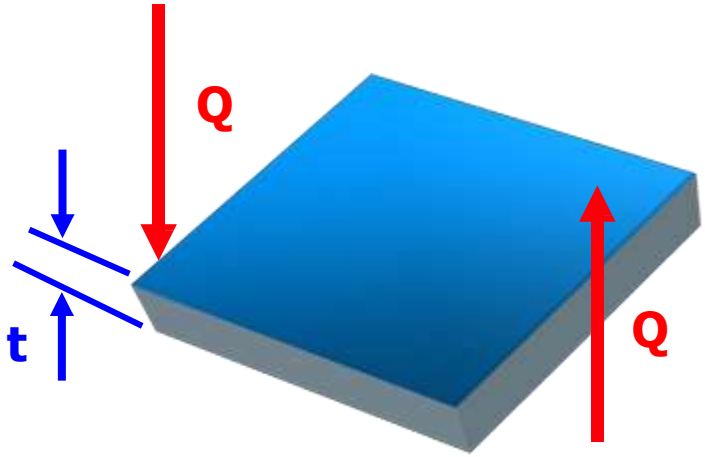
$$\{\tau\}^k = [\overline{Q}]_k \cdot \{\gamma\}^k = [\overline{Q}]_k \cdot \{\gamma^0\}$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Stresses in solid beams

## Interlaminar shear forces in beams

### Laminate equivalent constants




$$\gamma^0_{xz} = \frac{a_{55}}{K} \cdot Q \qquad \tau^o_{xz} = \frac{Q}{t}$$

In isotropic materials:

$$\tau_{xz} = G \cdot \gamma_{xz} \qquad \tau_{xz} = \frac{F}{A_s}$$

$$\Downarrow$$

$$G^0_{xz} = \frac{1}{a_{55} \cdot t} \qquad A_s = K \cdot w \cdot t$$

$$\langle GA_s \rangle = \frac{K \cdot w}{a_{55}}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Chapter 4. Composite beams and plates

# Composite beams

1. Introduction
2. Analysis of beams
3. Stresses in solid beams
4. **Displacements in solid beams**
5. Stresses in thin-walled beams
6. Displacements in thin-walled beams
7. References

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# Displacements in solid beams

- **Long beams**
  - L/H>10
  - The shear force contribution to the displacements can be neglected. Displacements are mainly produced by bending moment

$$d = d_b$$

$d_b$ controlled by EI

- **Short beams**
  - L/H<10
  - The shear force contribution to the displacements can not be neglected

$$d = d_b + d_s$$

$d_b$ controlled by EI

$d_s$ controlled by GA

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# Displacements in solid beams

## Calculation process

- Classical laminate theory is used to find equivalent stiffness to bending moments ($E_b$) and shear forces ($G_{xz}$)
- Navier-Bresse formulation can be used to find the displacements

$$\theta_B = \theta_A - \int_A^B \frac{M}{EI} \cdot ds$$

$$v_B = v_A + \theta_A \cdot (z_B - z_A) + \int_A^B \left( \frac{N}{EA} dy + \frac{Q}{GA_s} dZ \right) - \int_A^B \frac{M}{EI} (z_B - z) \cdot ds$$

$$w_B = w_A + \theta_A \cdot (y_B - y_A) + \int_A^B \left( \frac{N}{EA} dz - \frac{Q}{GA_s} dy \right) + \int_A^B \frac{M}{EI} (y_B - y) \cdot ds$$

- Where

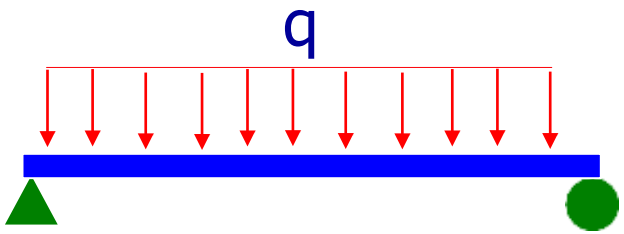
$E \rightarrow Ex_b$

$G \rightarrow G_{xz}$

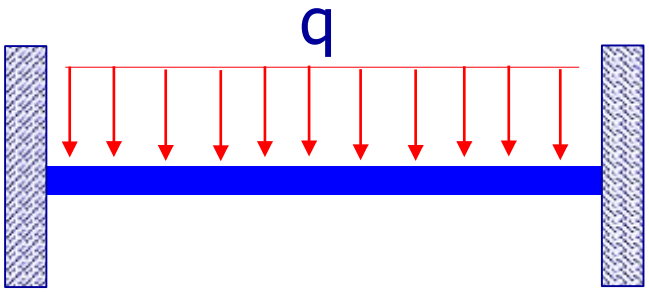
**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Displacements in solid beams

## Maximum displacements




$$\delta_{max} = \frac{5}{384}\frac{qL^4}{\langle EI \rangle} + \frac{1}{8}\frac{qL^2}{\langle GA_s \rangle}$$

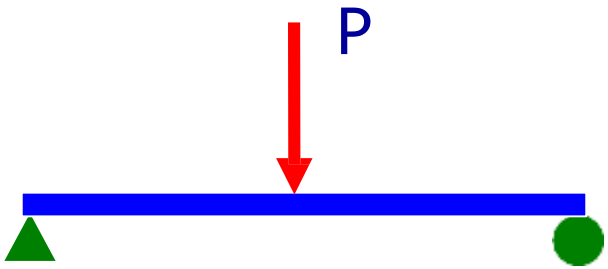



$$\delta_{max} = \frac{1}{384}\frac{qL^4}{\langle EI \rangle} + \frac{1}{8}\frac{qL^2}{\langle GA_s \rangle}$$

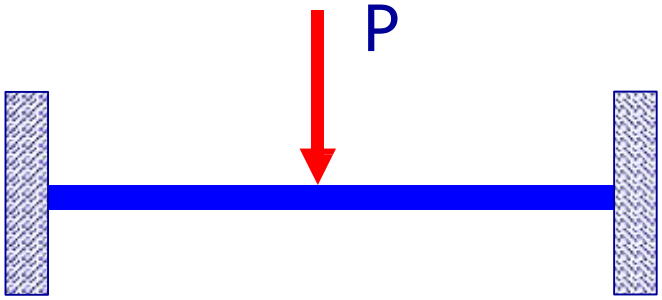
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Displacements in solid beams

## Maximum displacements




$$\delta_{max} = \frac{1}{48}\frac{PL^3}{\langle EI \rangle} + \frac{1}{4}\cancel{\frac{PL}{\langle GA_s \rangle}}$$




$$\delta_{max} = \frac{1}{192}\frac{PL^3}{\langle EI \rangle} + \frac{1}{4}\cancel{\frac{PL}{\langle GA_s \rangle}}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Displacements in solid beams

## Maximum displacements




$$\delta_{max} = \frac{ML^2}{8\langle EI \rangle}$$

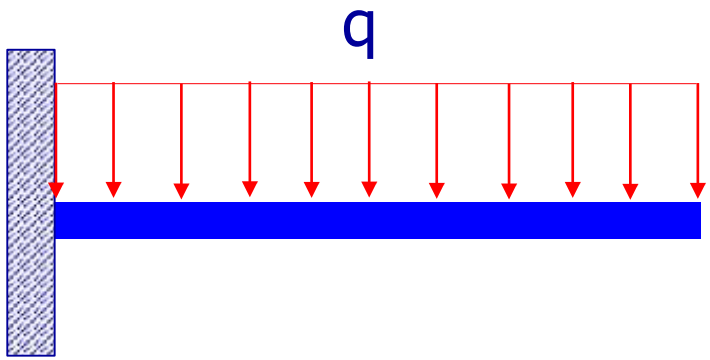



$$\delta_{max} = \frac{PL^3}{3\langle EI \rangle} + \cancel{\frac{PL}{\langle GA_s \rangle}}$$

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# Displacements in solid beams

## Maximum displacements



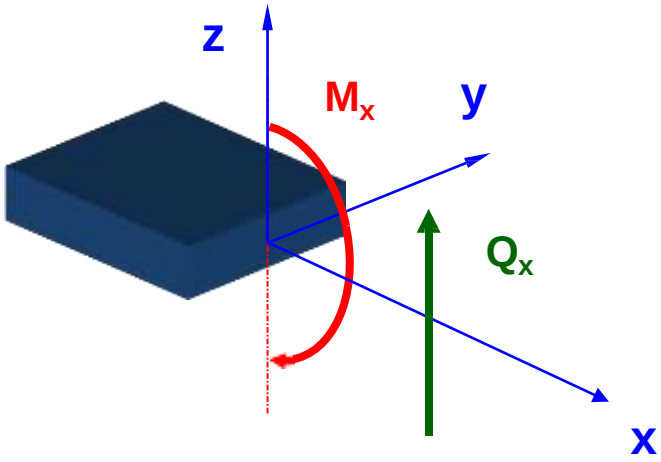

$$\delta_{max} = \frac{1}{8}\frac{qL^4}{\langle EI \rangle} + \cancel{\frac{1}{2}\frac{qL^2}{\langle GA_s \rangle}}$$

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# Displacements in solid beams

## Analysis of beams




$M_y = 0$

$M_{xy} = 0$

$Q_y = 0$

$\phi_y = 0$

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# Displacements in solid beams

## Analysis of beams

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{1s} \\ D & D & D \\ D_{1s}^{12} & D_{2s}^{22} & D_{ss}^{2s} \end{bmatrix} \cdot \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$$

$$\{k\} = [d]\{M\} \quad \Longleftrightarrow$$

$$k_x = \frac{\partial \phi_x}{\partial x}$$

$$k_y = \frac{\partial \phi_y}{\partial y}$$

$$k_{xy} = \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Displacements in solid beams

## Analysis of beams

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \cdot \begin{bmatrix} A_{44} & A_{45} \\ A_{54} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

$$\{\gamma\} = \frac{1}{K}\, a\, [Q]\{\}$$

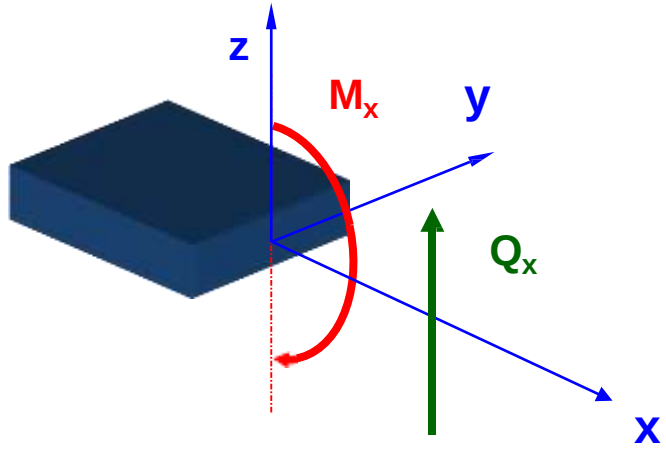
$$\gamma_{yz} = \frac{\partial w}{\partial y} + \phi_y$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \phi_x$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Displacements in solid beams

## Analysis of beams




$$\frac{\partial \phi_x}{\partial x} = d_{11} M_x$$

$$\frac{\partial w}{\partial x} + \phi_x = \frac{a_{55}}{K} Q_x$$

$M_y = 0 \quad Q_y = 0$

$M_{xy} = 0 \quad \phi_y = 0$

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# Chapter 4. Composite beams and plates

# Composite beams

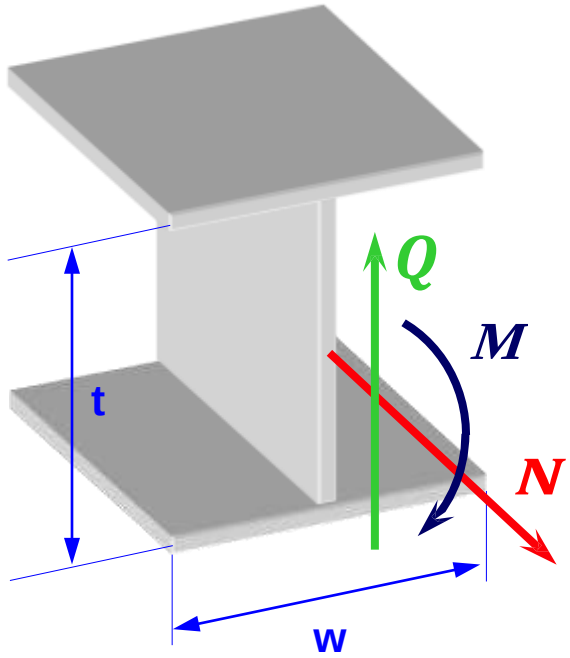
1. Introduction
2. Analysis of beams
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4. Displacements in solid beams
5. **Stresses in thin-walled beams**
6. Displacements in thin-walled beams
7. References

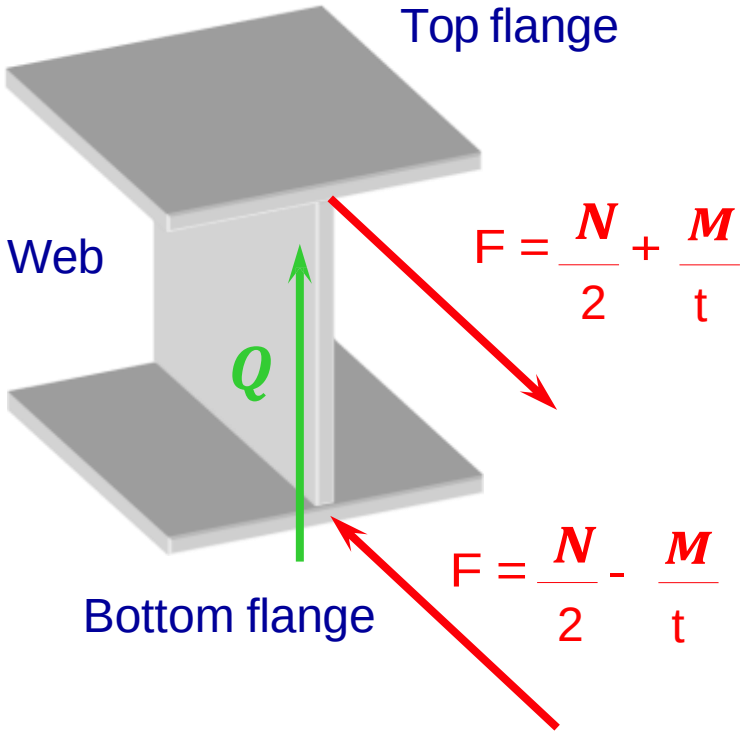
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# Stresses in thin-walled beams

## Bending of beams

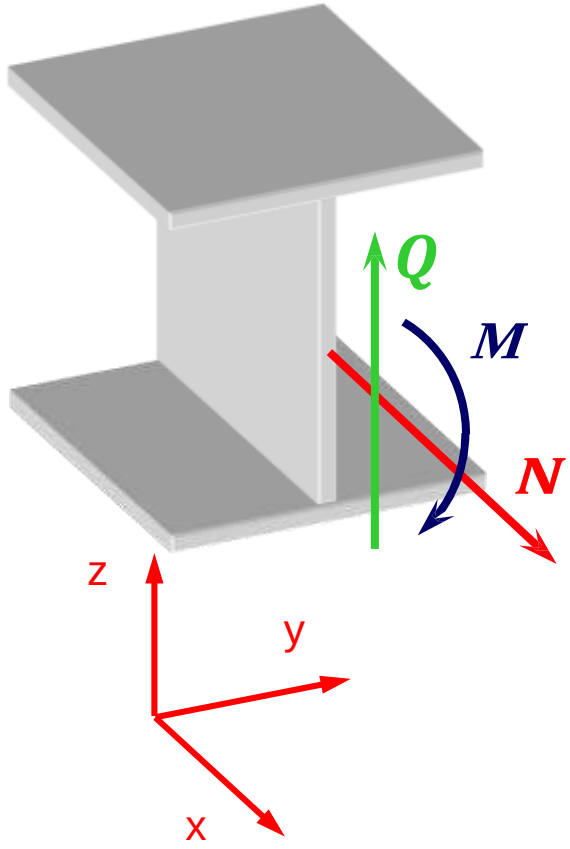


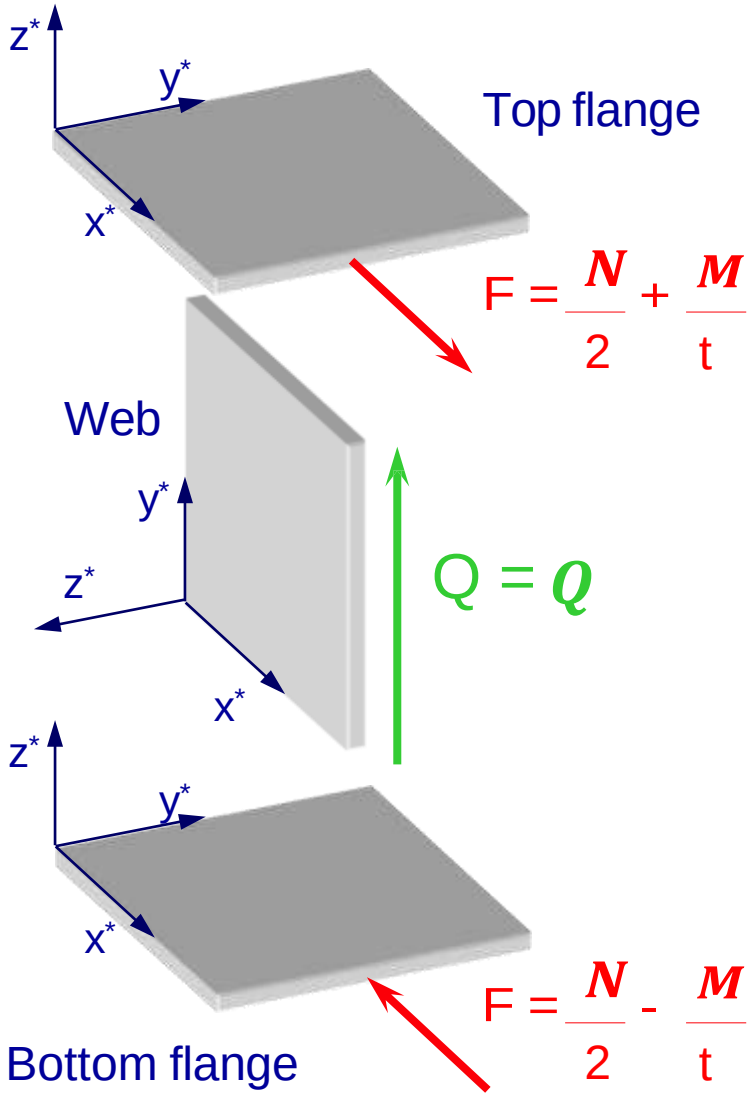




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# Stresses in thin-walled beams

## Bending of beams




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# Stresses in thin-walled beams

## Bending of beams

Top flange

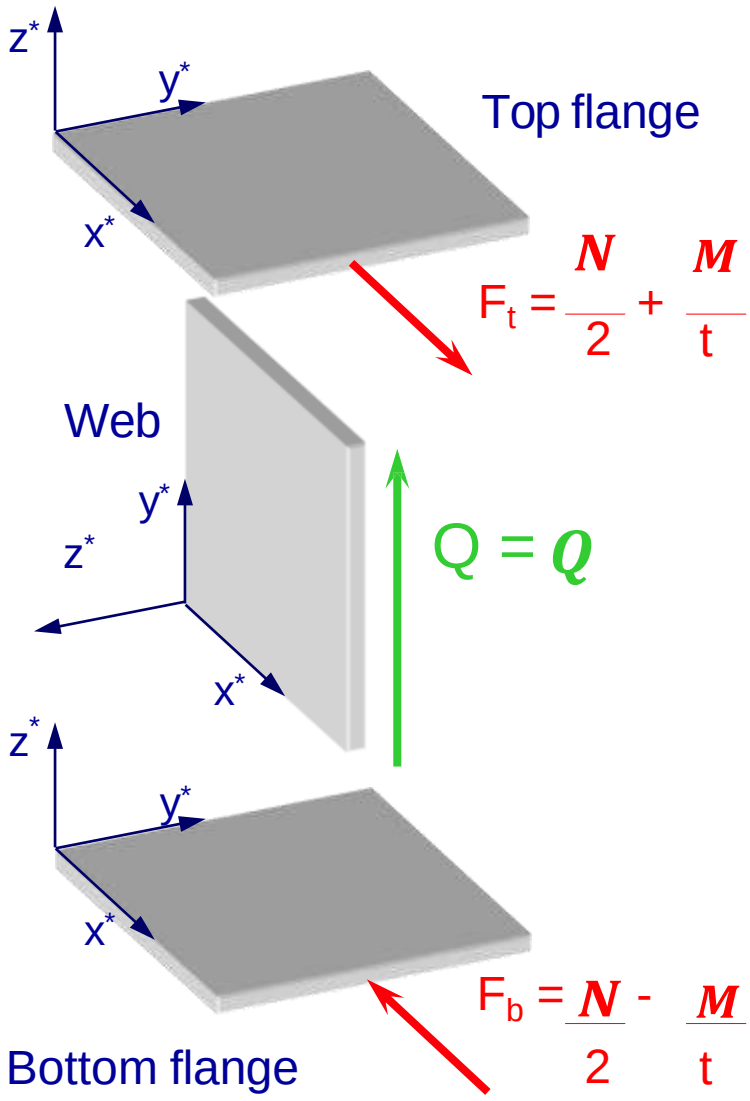
$$\{N\} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{Bmatrix} F_t / w \\ 0 \\ 0 \end{Bmatrix}$$

Web

$$\{N\} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q / t \end{Bmatrix}$$

Bottom flange

$$\{N\} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{Bmatrix} F_b / w \\ 0 \\ 0 \end{Bmatrix}$$



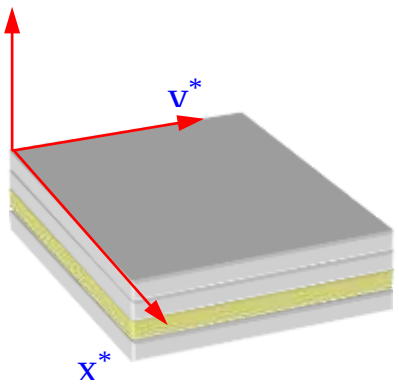

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# Stresses in thin-walled beams

## Bending of beams

**Load state**

$\{N\}$



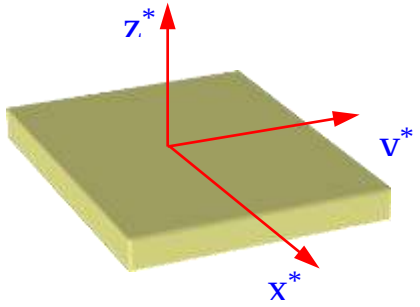

**Strains**

$$\{N\} = [A] \cdot \{\varepsilon^0\}$$

**Stresses field**

(In each layer)

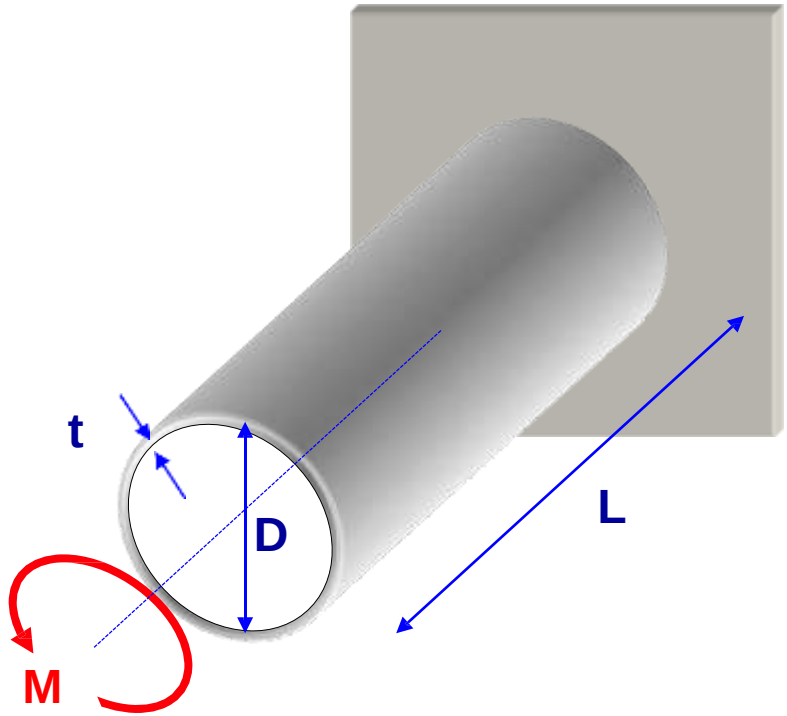
$$\{\sigma\}^{xy}_i = [\bar{Q}]_i \left( \{\varepsilon^0\} + z \cdot \{k\} \right)$$




**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Stresses in thin-walled beams

## Torsion of beams




**Hypothesis:**

**H<<D**

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Stresses in thin-walled beams

## Torsion of beams

### Internal forces

Theory for isotropic materials:

$$\tau_{xy}^{0} = \frac{2 \cdot M}{\pi \cdot D^2 \cdot t}$$

$$\Rightarrow \quad \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{2 \cdot M}{D \cdot t} \end{Bmatrix}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Stresses in thin-walled beams

## Torsion of beams

### Strains

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{1s} \\ a_{12} & a_{22} & a_{22} \\ a_{1s} & a_{22} & a_{ss} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dfrac{2 \cdot M}{D \cdot t} \end{Bmatrix}$$

En particular $$\gamma_{xy}^o = a_{ss} \cdot \frac{2 \cdot M}{D \cdot t}$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Stresses in thin-walled beams

## Torsion of beams

**Strains**

$$\gamma_{xy}^{o} = a_{ss} \cdot \frac{2 \cdot M}{D \cdot t}$$

$$\gamma_{xy}^{0} = \frac{2 \cdot M}{\pi \cdot D^2 \cdot t \cdot G_{xy}^{0}} \qquad G_{xy}^{0} = \frac{\pi \cdot D}{a_{ss}}$$

**Rotations:**

$$\theta_{\max} = \frac{M \cdot L}{G_{xy}^{0} \cdot I_o} \qquad I_o = \frac{\pi \cdot D^3 \cdot t}{4}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

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# Displacements in thin-walled beams

## Chapter 4. Composite beams and plates

## Composite beams

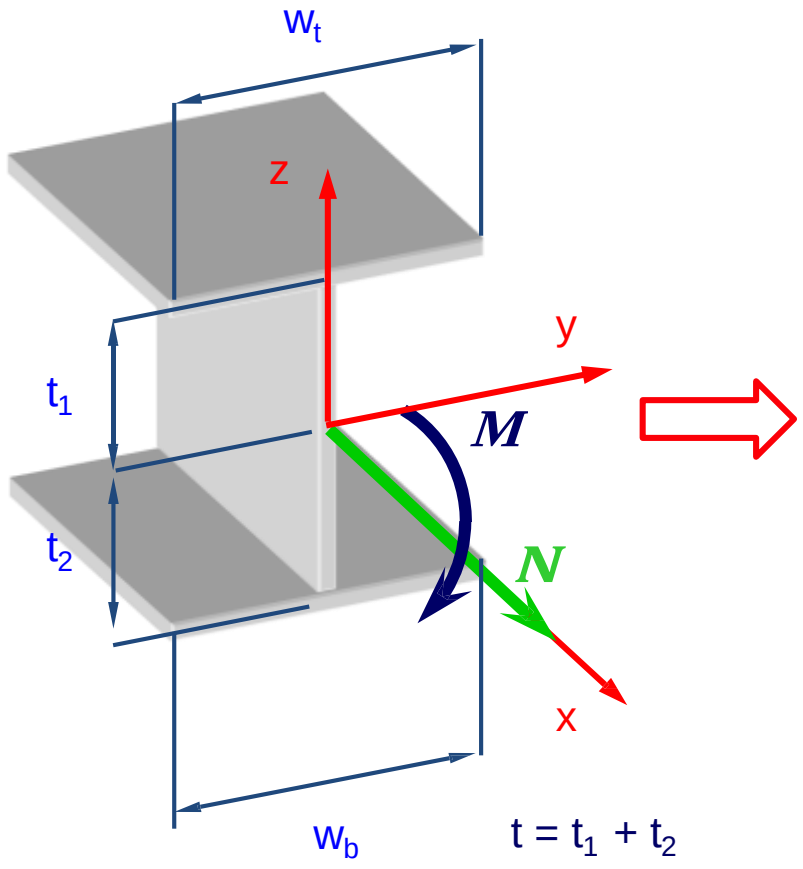
1. Introduction
2. Analysis of beams
3. Stresses in solid beams
4. Displacements in solid beams
5. Stresses in thin-walled beams
6. **Displacements in thin-walled beams**
7. References

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# Displacements in thin-walled beams

## Axial force and bending moment




### Hypotheses

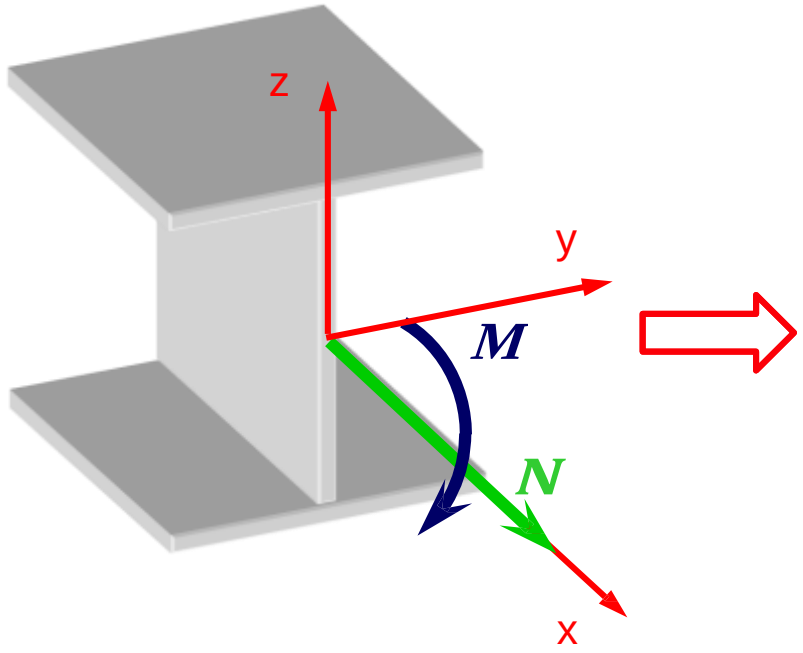
- The bending moment can be found through the integral of the normal stress along the z direction.
- Longitudinal strain in x direction is directly related to the distance from neutral axis.

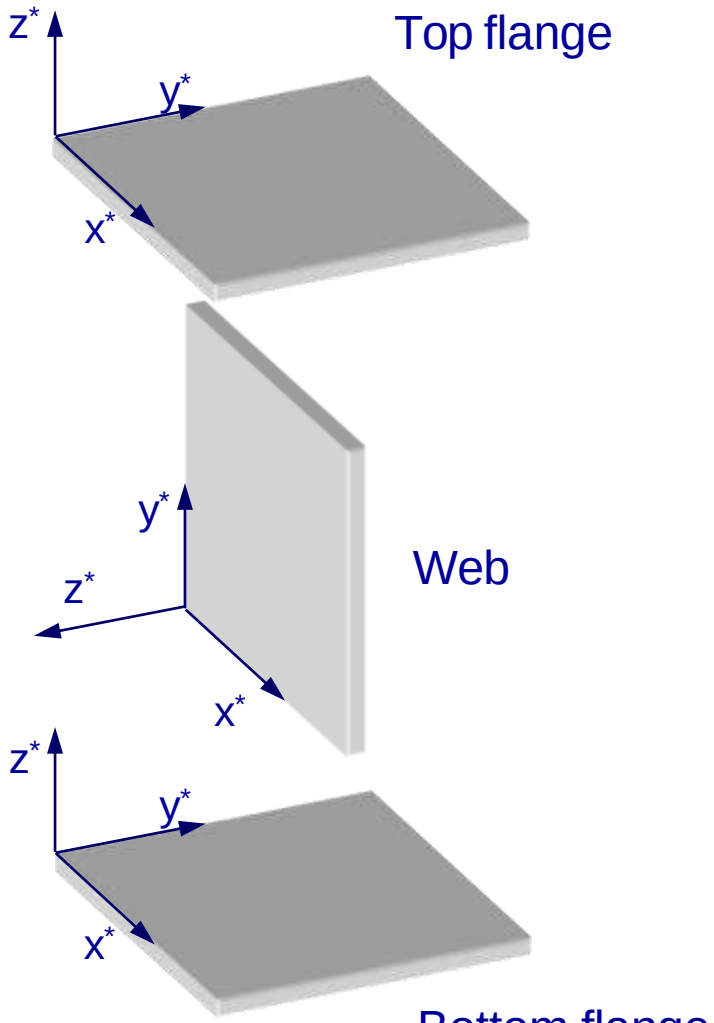
$$\varepsilon_x(z) = z \cdot k_x$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Displacements in thin-walled beams

## Axial force and bending moment



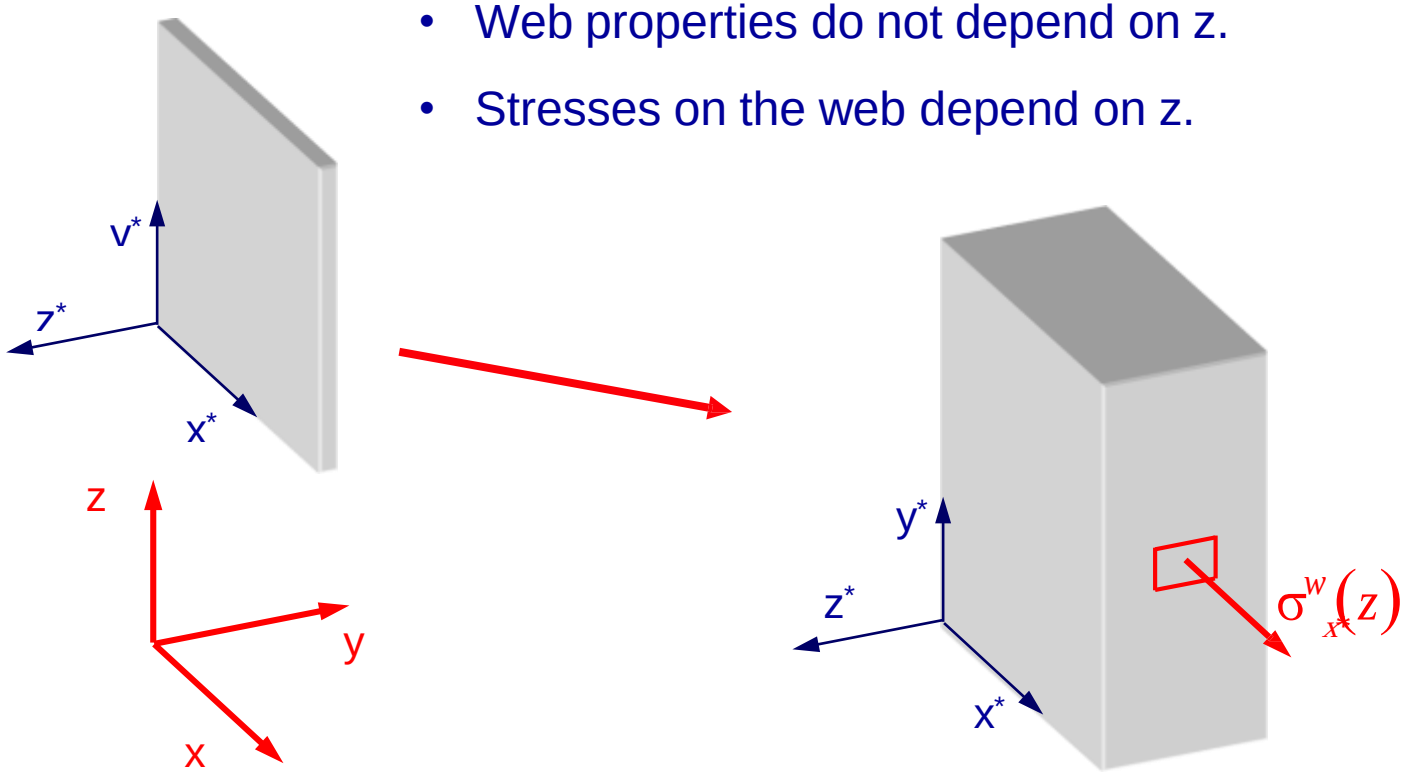

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Displacements in thin-walled beams

## Axial force and bending moment

**Web**

- Bending stiffness matrix [D] is not necessary.
- Web properties do not depend on z.
- Stresses on the web depend on z.



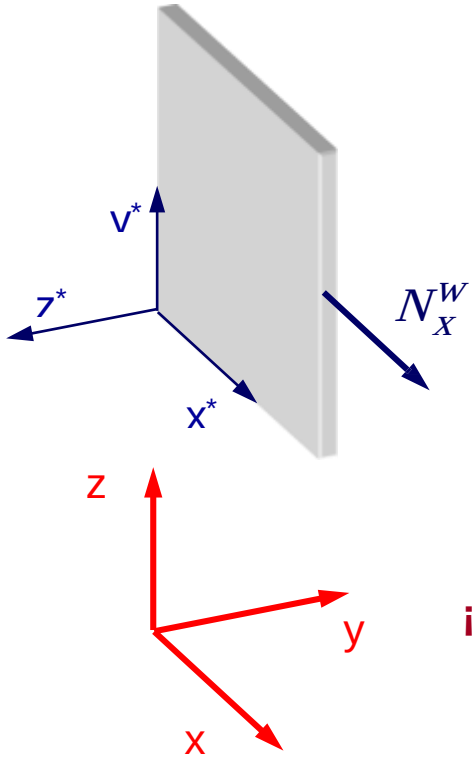

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# Displacements in thin-walled beams

## Axial force and bending moment

**Web**




$$N_{x^*}^{w} = \int \sigma_{x^*}^{w} \cdot dz^*$$

$$\begin{Bmatrix} \varepsilon^{o^w}_{x^*} \\ \varepsilon^{o^w}_{y^*} \\ \gamma^{w}_{xy^*} \end{Bmatrix} = \begin{bmatrix} a^w_{11^*} & a^w_{11^*} & a^w_{11^*} \\ a^w_{11^*} & a^w_{11^*} & a^w_{11^*} \\ a^w_{11^*} & a^w_{11^*} & a^w_{11^*} \end{bmatrix} \cdot \begin{Bmatrix} N^w_{x^*} \\ 0 \\ 0 \end{Bmatrix}$$

$$\varepsilon^{o^w}_{x^*} = a^w_{11^*} \cdot N^w_{x^*}$$

$$\varepsilon^{o^w}_{x}(z) = a^w_{11^*} \cdot N^w_{x}(z)$$

¡ Axes x* and x match up !

¡ Results depend on global coordinate z!

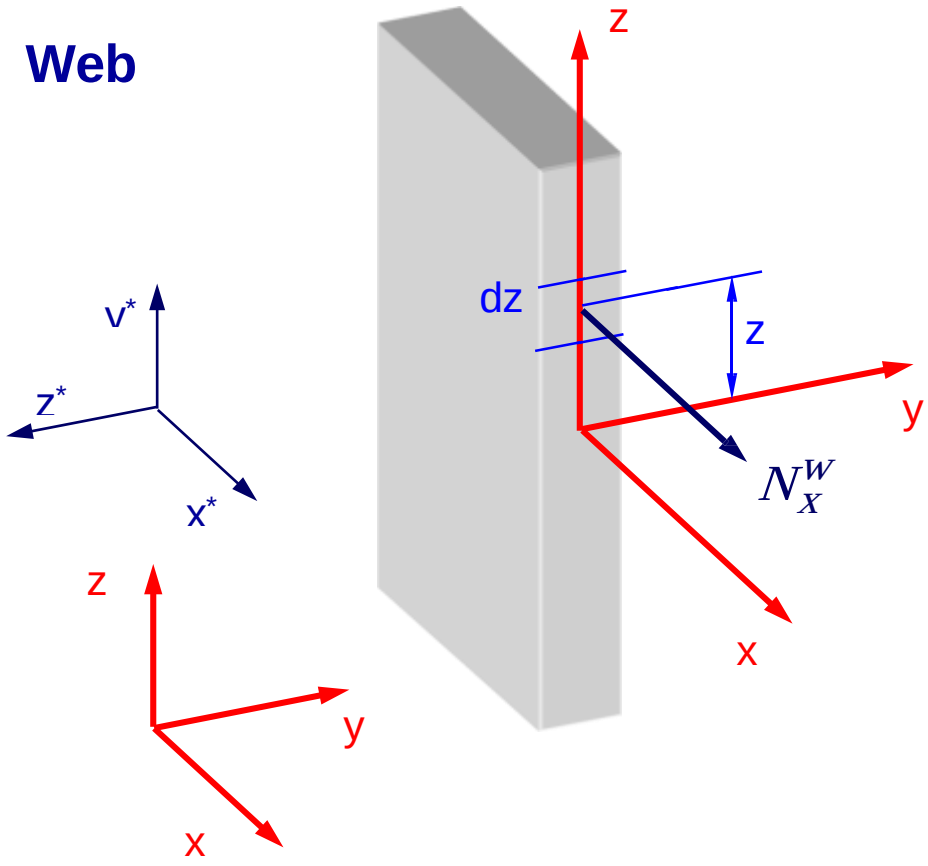
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# Displacements in thin-walled beams

## Axial force and bending moment

**Web**




$$\varepsilon_x^{o\,w}(z) = z \cdot k_x$$

$$N_x^w = \frac{1}{a_{11*}^w} \cdot z \cdot k_x$$

**Axial force per unit length, coordinate y* (coordinate z)**

This axial force contributes to the section bending moment

$$M_w = \int_{-t/2}^{t/2} N_x^w \cdot z \cdot dz$$

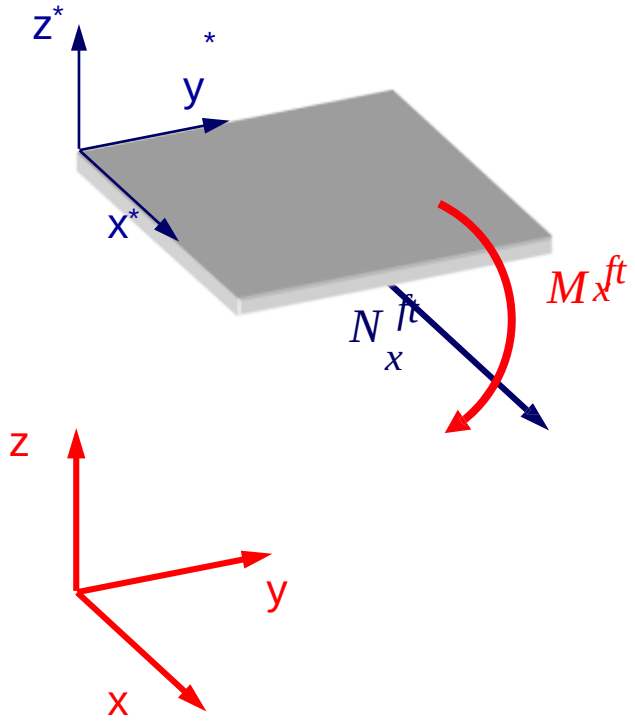
$$M_w = \frac{t^3}{12 \cdot a_{11*}^w} \cdot k_x$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Displacements in thin-walled beams

## Axial force and bending moment

### Top flange




$$\begin{Bmatrix} \varepsilon_{x^*}^{o\,ft} \\ \varepsilon_{y^*}^{o\,ft} \\ \gamma_{xy^*}^{o\,ft} \end{Bmatrix} = \begin{bmatrix} a_{11^*}^{ft} & a_{11^*}^{ft} & a_{11^*}^{ft} \\ a_{11^*}^{ft} & a_{11^*}^{ft} & a_{11^*}^{ft} \\ a_{11^*}^{ft} & a_{11^*}^{ft} & a_{11^*}^{ft} \end{bmatrix} \cdot \begin{Bmatrix} N_{x^*}^{ft} \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} k_{x^*} \\ k_{y^*} \\ k_{xy^*} \end{Bmatrix} = \begin{bmatrix} d_{11^*}^{ft} & d_{11^*}^{ft} & d_{11^*}^{ft} \\ d_{11^*}^{ft} & d_{11^*}^{ft} & d_{11^*}^{ft} \\ d_{11^*}^{ft} & d_{11^*}^{ft} & d_{11^*}^{ft} \end{bmatrix} \cdot \begin{Bmatrix} M_{x^*}^{ft} \\ 0 \\ 0 \end{Bmatrix}$$

$$\varepsilon_{x^*}^{o\,ft} = a_{11^*}^{ft} \cdot N_{x^*}^{ft}$$
$$k_{x^*} = d_{11^*}^{ft} \cdot M_{x^*}^{ft}$$

$$\Rightarrow \quad \varepsilon_x^{o\,ft} = a_{11^*}^{ft} \cdot N_x^{ft}$$
$$k_x = a_{11^*}^{ft} \cdot M_x^{ft}$$

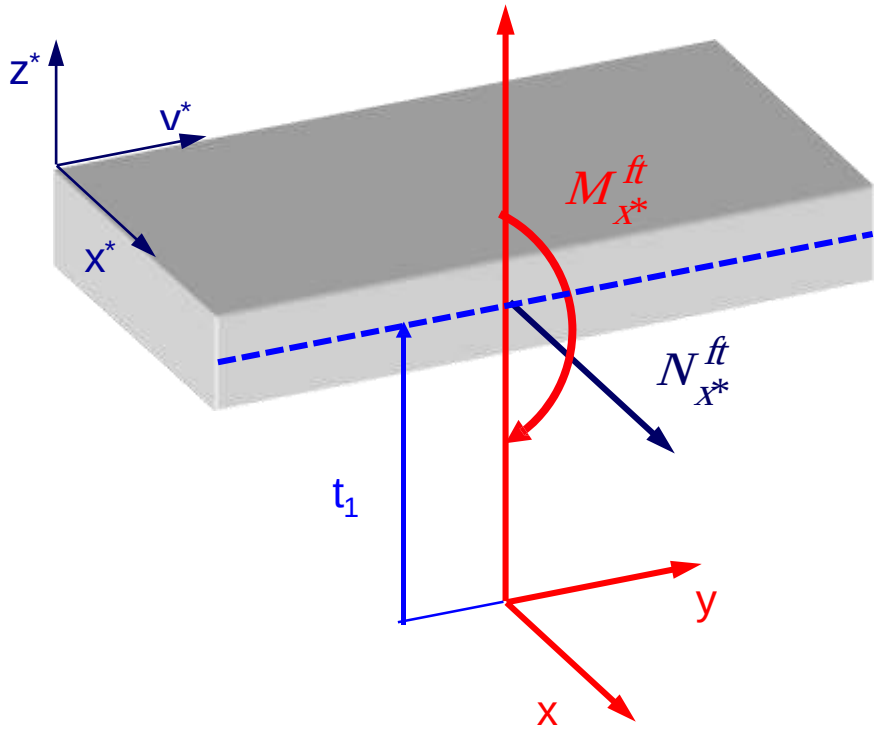
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# Displacements in thin-walled beams

## Axial force and bending moment

### Top flange




Axial force and bending moment in the flange contribute to the section bending moment

$$\boldsymbol{M}_{\boldsymbol{ft}} = w_t \cdot \left( N^{ft}_x \cdot t_1 + M^{ft}_x \right)$$

$$\boldsymbol{M}_{\boldsymbol{ft}} = w_t \cdot \left( \frac{t_1}{a^{ft}_{11^*}} \cdot \varepsilon^{o\,ft}_x + \frac{1}{d^{ft}_{11^*}} \cdot k_x \right)$$

Strains at flange mid-plane

$$\varepsilon^{o\,ft}_x = t_1 \cdot k_x$$

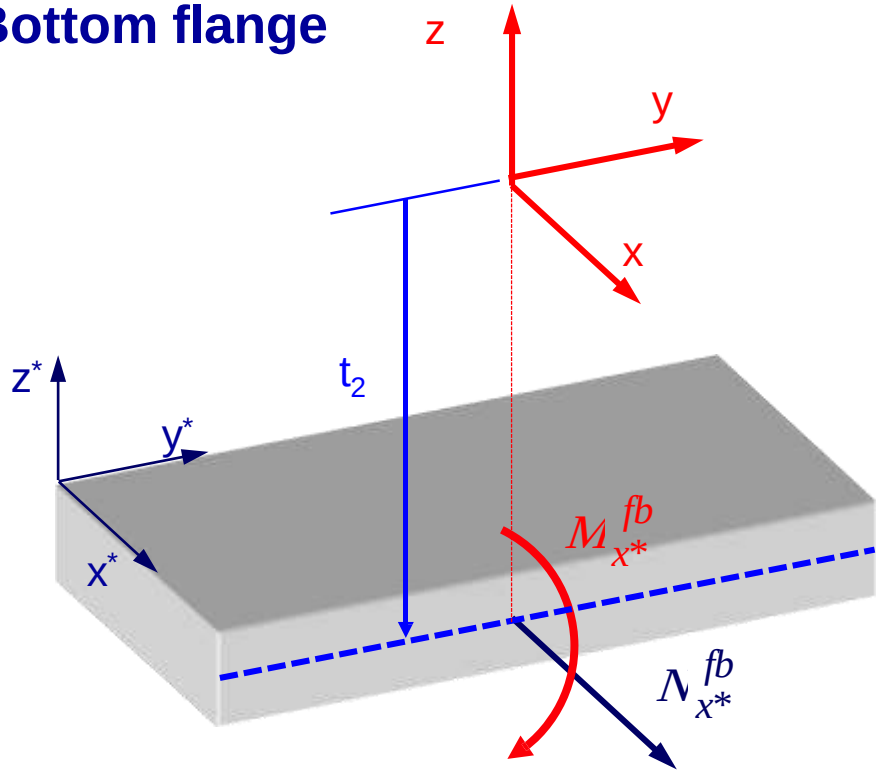
$$\boldsymbol{M}_{\boldsymbol{ft}} = w_t \cdot \left( \frac{t_1^2}{a^{ft}_{11^*}} \cdot + \frac{1}{d^{ft}_{11^*}} \right) \cdot k_x$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Displacements in thin-walled beams

## Axial force and bending moment

### Bottom flange




By analogy:

$$M_{fb} = w_b \cdot \left( \frac{t_2^2}{a^{fb}_{11^*}} \cdot + \frac{1}{d^{fb}_{11^*}} \right) \cdot k_x$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Displacements in thin-walled beams

## Axial force and bending moment

The section bending moment is the sum of the bending moments supported by web and flanges:

$$\boldsymbol{M} = \left[ w_t \cdot \left( \frac{t_1^2}{a_{11*}^{ft}} \cdot + \frac{1}{d_{11*}^{ft}} \right) + w_b \cdot \left( \frac{t_2^2}{a_{11*}^{fb}} \cdot + \frac{1}{d_{11*}^{fb}} \right) + \frac{(t_1 + t_2)^3}{12 \cdot a_{11*}^{w}} \right] \cdot k_x$$

In isotropic material beams:

$$\boldsymbol{M} = EI \cdot k_x$$

$$\langle EI \rangle = w_t \cdot \left( \frac{t_1^2}{a_{11*}^{ft}} \cdot + \frac{1}{d_{11*}^{ft}} \right) + w_b \cdot \left( \frac{t_2^2}{a_{11*}^{fb}} \cdot + \frac{1}{d_{11*}^{fb}} \right) + \frac{(t_1 + t_2)^3}{12 \cdot a_{11*}^{w}}$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# References

## Chapter 4. Composite beams and plates

## Composite beams

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5. Stresses in thin-walled beams
6. Displacements in thin-walled beams
7. References

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

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# References

**The Behaviour of Structures Composed of Composite Materials**

J.R. Vinson y R.L. Sierakowski

Kluwer Academic Publishers. 2002.

Cap. 4: Beams, columns and Rods of Composite Materials

**Mechanics of laminated composite plates. Theory and analysis**

J.N. Reddy

CRC Press. 1997.

Cap 6: One- Dimensional Analysis of Laminated Plates




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# Aerospace Structures

## Chapter 4. Laminate and sandwich structures

### Sandwich structures

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6. References

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Chapter 4. Composite beams and plates

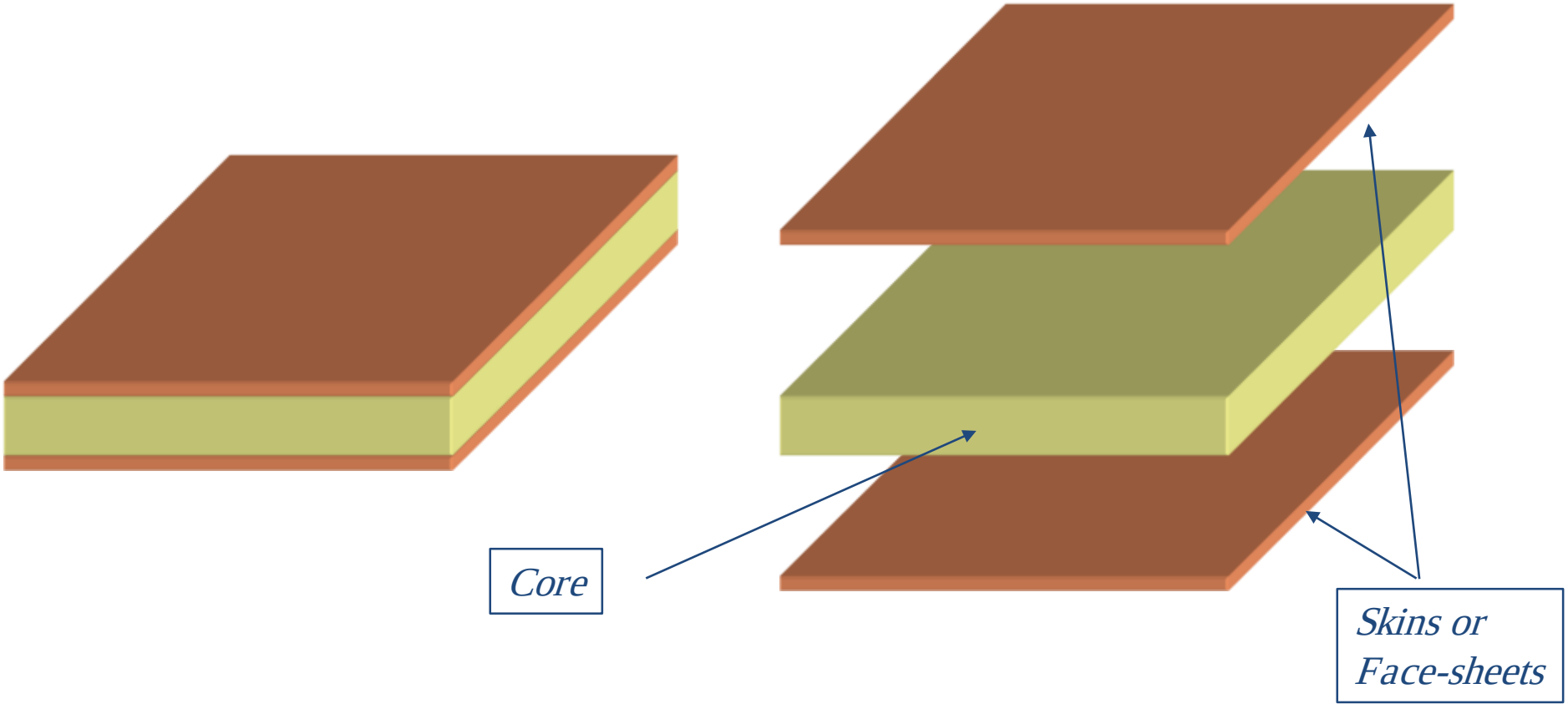
# Sandwich structures

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Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Introduction

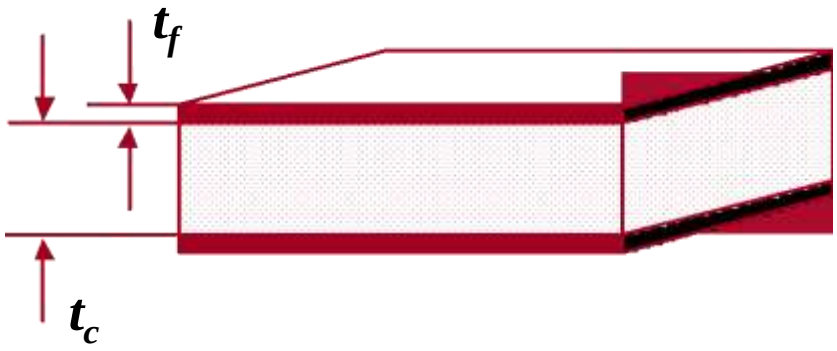
## Definition of a sandwich structure




Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Introduction

## Sandwich parameters




Geometry

$$10 \leq \frac{t_c}{t_f} \leq 100$$

$$0.25\ mm \leq e_f \leq 12.7\ mm$$

Core density

$$20\ \frac{kg}{m^3} \leq \rho_c \leq 1000\ \frac{kg}{m^3}$$

Adhesive thickness

$$0.025\ mm \leq t_a \leq 0.2\ mm$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Introduction

## Advantages of sandwich

- **High strength and stiffness under bending moments**




| Core thickness | 0 | t | 3·t |
|---|---|---|---|
| Bending stiffness | 1 | 7 | 37 |
| Bending strength | 1 | 3,5 | 9,25 |
| Weight | 1 | 1,03 | 1,06 |

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Introduction

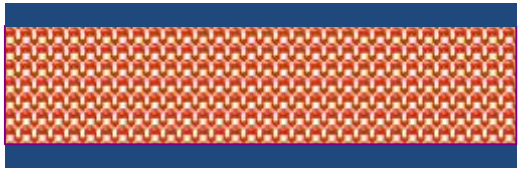
## Advantages of sandwich

- **Thermal isolation**



2 mm

$$\overset{o}{Q} = 1 \text{ MW}$$



1 mm

5 mm

1 mm

$$\overset{o}{Q} = 300 \text{ W}$$

Metal: $k = 100 \frac{W}{m\,^{o}C}$

Face-sheet: $k = 100 \frac{W}{m\,^{o}C}$

PUR core: $k = 0{,}07 \frac{W}{m\,^{o}C}$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

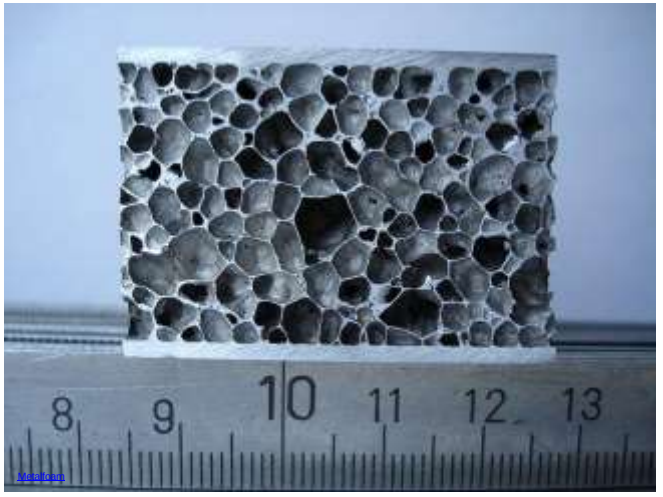
# Introduction

## Core typologies



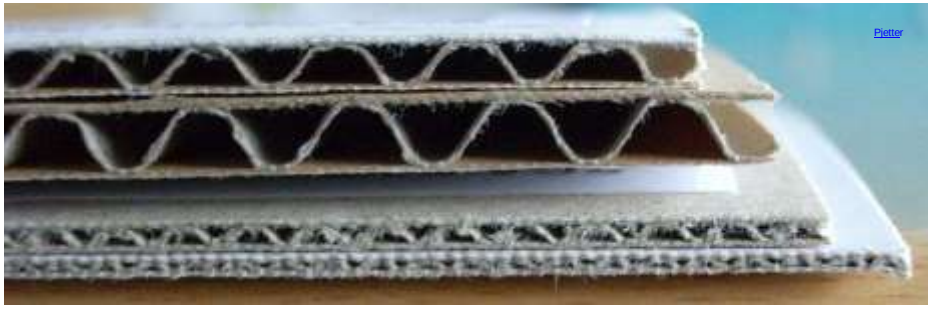

http://commons.wikimedia.org/wiki/File:Steinbichler_Shearography_Honeycomb_with_CFRP_Top_Layer_Artificial_failures_that_simulate_layer-core_delaminations_Material.jpg?uselang=es

Honeycomb panel




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Foams



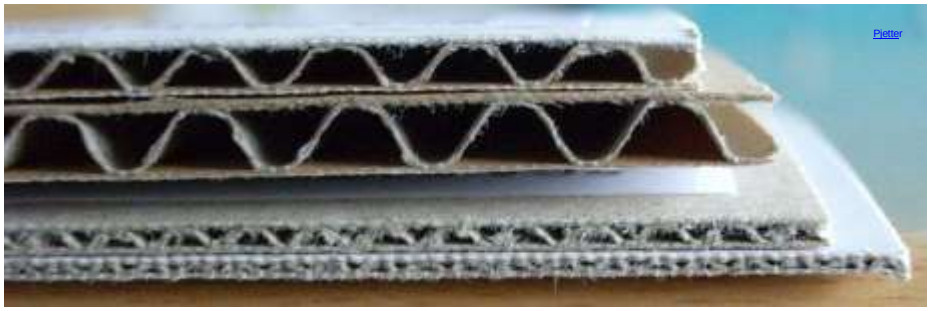

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Corrugated core

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Introduction

## Core typologies




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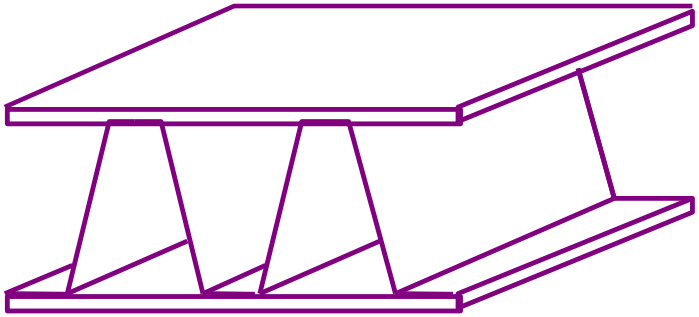
Corrugated core

**Face-sheets**

Metals
Laminates

**Core**

Metals
Laminates



**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

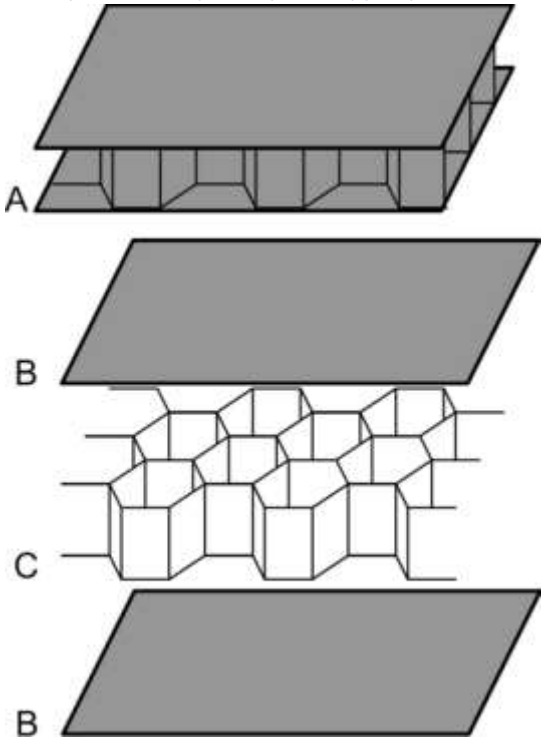
# Introduction

## Core typologies




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George William Herbert
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**Face-sheets**

Metals
Laminates

**Core**

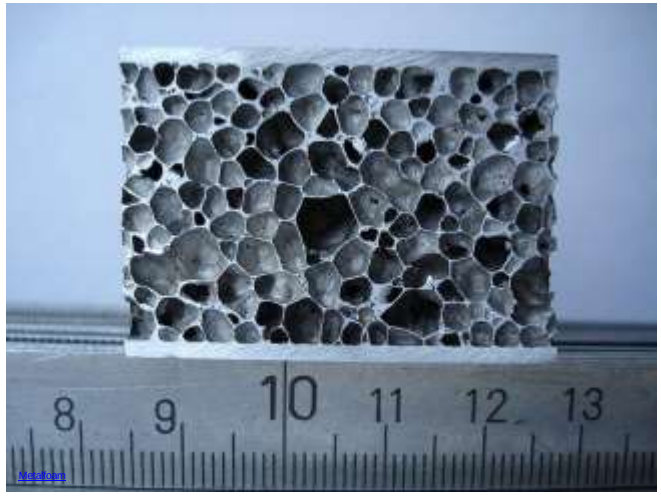
Aluminium alloy (Al5052, Al5056, Al2024)
Nomex® (aramid fibres/ Phenol formaldehyde resin)
Kraft Cardboard
Carbon-epoxy
Kevlar-epoxy

Honeycomb panel

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Introduction

## Core typologies




http://commons.wikimedia.org/wiki/File:Aluminium_foam_sandwich.jpg?uselang=es

Foams

**Face-sheets**

Metals
Laminates
Plywood

**Core**

Phenol formaldehyde resin
PVC
Metallic foams

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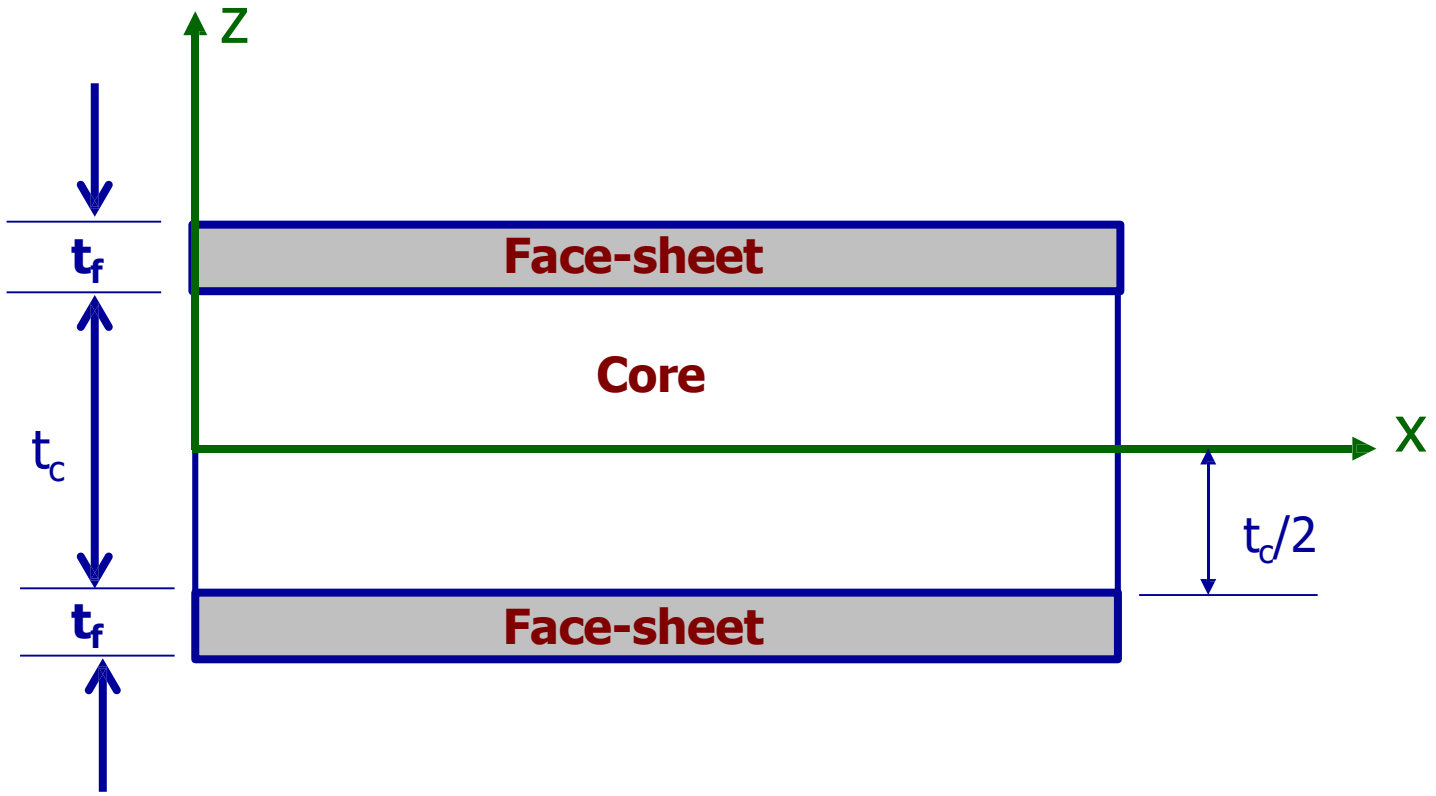
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# Hypotheses




L = Length

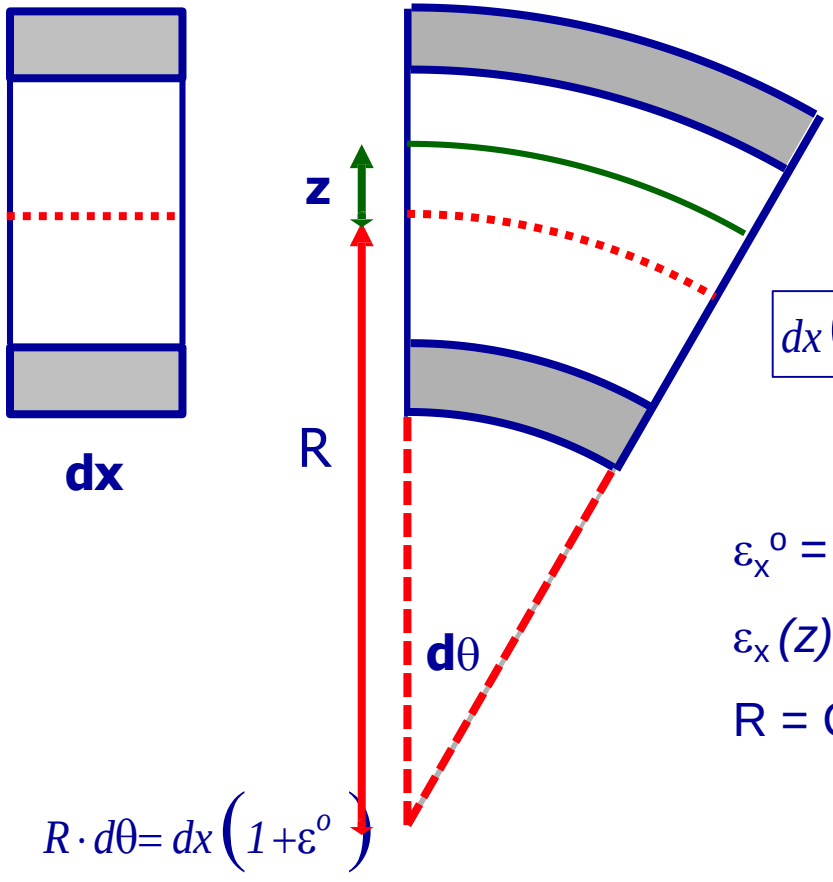
w = Width

- Hypotheses:
  - $t_c >> t_f$
  - $E_c << E_f$

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# Hypotheses




$\varepsilon_x{}^o$ = Mid-plane strain

$\varepsilon_x(z)$ = Strain at a distance $z$ from mid-plane

R = Curvature radius

$$R\cdot d\theta = dx\left(1+\varepsilon^o_x\right)$$

$$(R+z)d\theta = dx\left(1+\varepsilon_x(z)\right)$$

$$\Rightarrow \quad \varepsilon_x(z) = \varepsilon^o_x + \frac{z}{R}\left(1+\varepsilon^o_x\right) \quad \Rightarrow \quad \varepsilon_x(z) \cong \varepsilon^o_x + \frac{z}{R}$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Stress field in sandwich beams

## Chapter 4. Composite beams and plates

## Sandwich structures

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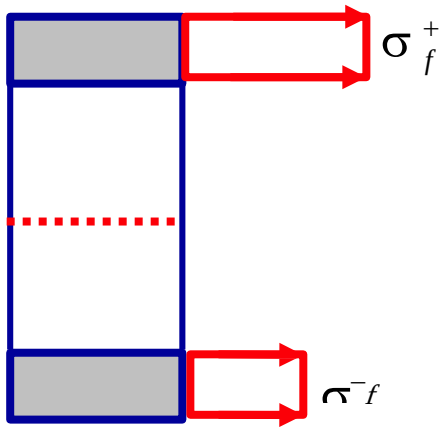
# Stress field in sandwich beams

## Normal stresses

Considering only normal stresses in x direction:

$$\sigma_x(z) \approx E(z)\cdot\left(\varepsilon_x^o + \frac{z}{R}\right)$$

Different Young modulus in core and face-sheets

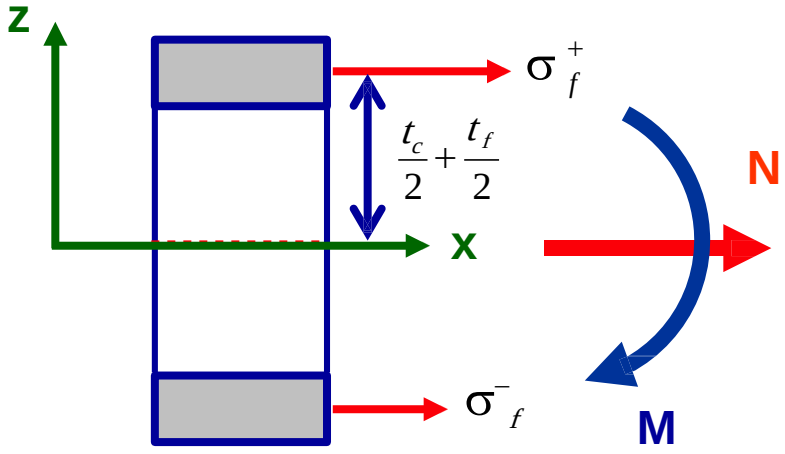



**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Stress field in sandwich beams

## Normal stresses

- Axial forces and bending moments




Internal forces can be found integrating normal stresses through the section

$$N = \sigma^+ t_f w + \sigma^- t_f w$$

$$M = \sigma_f^+ t_f w\left(\frac{t_f + t_c}{2}\right) - \sigma_f^- t_f w\left(\frac{t_f + t_c}{2}\right)$$

$$\sigma_f^+ = \frac{1}{2}\left(\frac{N}{t_f w} + \frac{2M}{t_f w(t_f + t_c)}\right)$$

$$\sigma_f^- = \frac{1}{2}\left(\frac{N}{t_f w} - \frac{2M}{t_f w(t_f + t_c)}\right)$$

**The units of these internal forces are:**

**N(N) and M(N·m)**

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Stress field in sandwich beams

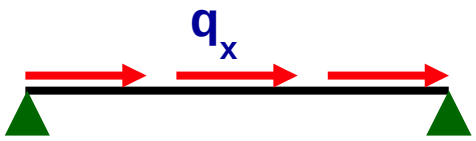
## Shear stresses

Equilibrium equation in direction x:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + X = 0 \quad \Rightarrow \quad \frac{\partial \tau_{xz}}{\partial z} = -X - \frac{\partial \sigma_x}{\partial x}$$

$X$ = Forces per unit volume

$q_x$= Forces per unit length in direction X




$$X = \frac{q_x}{w \cdot 2t_f}$$

Force per unit length, $q_x$; is uniformly distributed in the face-sheets section

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Stress field in sandwich beams

## Shear stresses

Force per unit length, $q_x$; is uniformly distributed in the face-sheets section

- In the core:

$$\left.\begin{array}{l} X = 0 \\ \sigma_x = 0 \end{array}\right\} \rightarrow \frac{\partial \tau_{xz}}{\partial z} = 0 \rightarrow \tau_{xz} = \tau_c = cte$$

- In the skins:

$$\left.\begin{array}{l} X = \dfrac{q_x(x)}{2wt_f} \\ \sigma_x = \sigma_f(x) \end{array}\right\} \rightarrow \frac{\partial \tau_{xz}}{\partial z} = f(x) \rightarrow \tau_{xz} \text{ directly related to z}$$

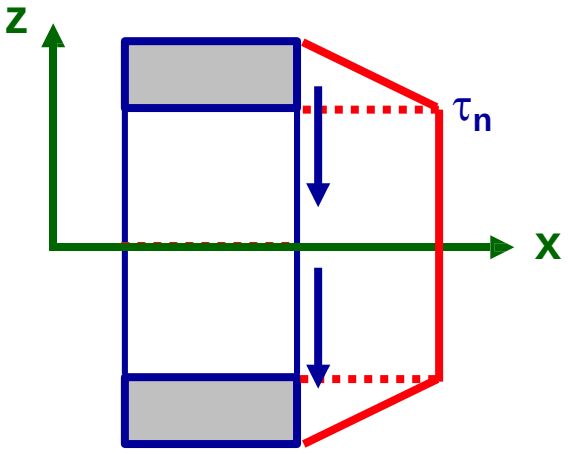
To verify the equilibrium equations there must be a discontinuity the shear stresses $\tau_{xz} = \tau_c$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

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# Stress field in sandwich beams

## Shear stresses




Equilibrium equation for bottom face-sheet:

$$\frac{\partial \tau_{xz}}{\partial z} = -\frac{\partial \bar{\sigma}_1}{\partial x} - \frac{q_x}{2wt_f}$$

Boundary conditions:

$$z = -\left(\frac{t_c}{2} + t_f\right) \rightarrow \tau_{xz} = 0$$

$$z = -\frac{t_c}{2} \rightarrow \tau_{xz} = \tau_c$$

$$\tau_c = -t_f\left(\frac{\partial \bar{\sigma}_f}{\partial x} + \frac{q_x}{2wt_f}\right)$$

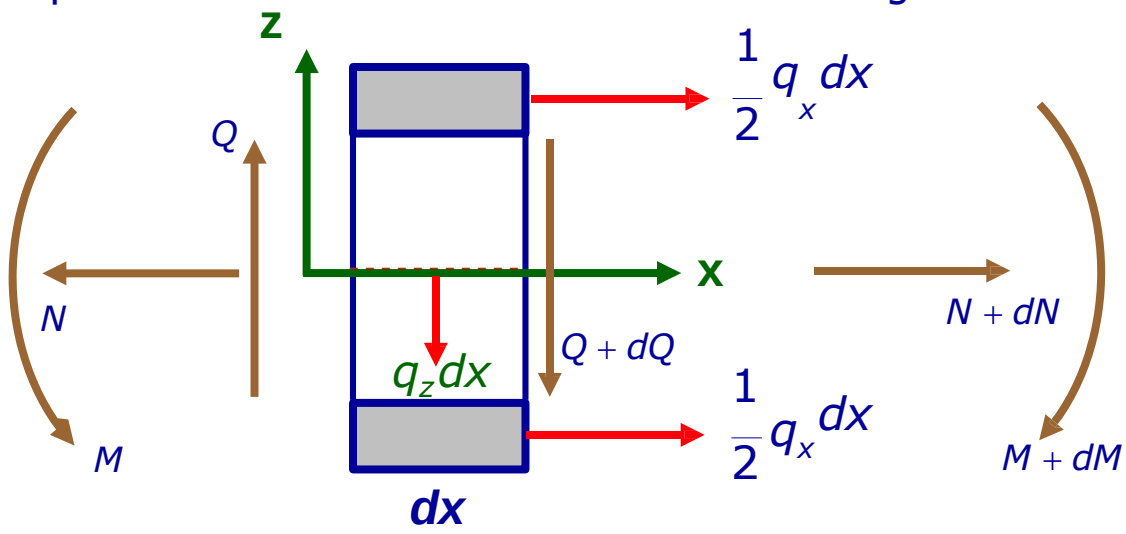
$$\tau_c = \frac{1}{2w}\left(\frac{\partial \Lambda}{\partial x} + \frac{2}{(t_1 + t_c)}\frac{\partial M}{\partial x} - q_x\right)$$

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# Stress field in sandwich beams

## Shear force

Equilibrium equations of a beam element of differential length dx




- Longitudinal forces equilibrium

$$-N+(N+dN)+q_x dx=0$$

$$\frac{dN}{dx}=-q_x$$

- Transverse forces equilibrium

$$-Q+(Q+dQ)+q_Z dx=0$$

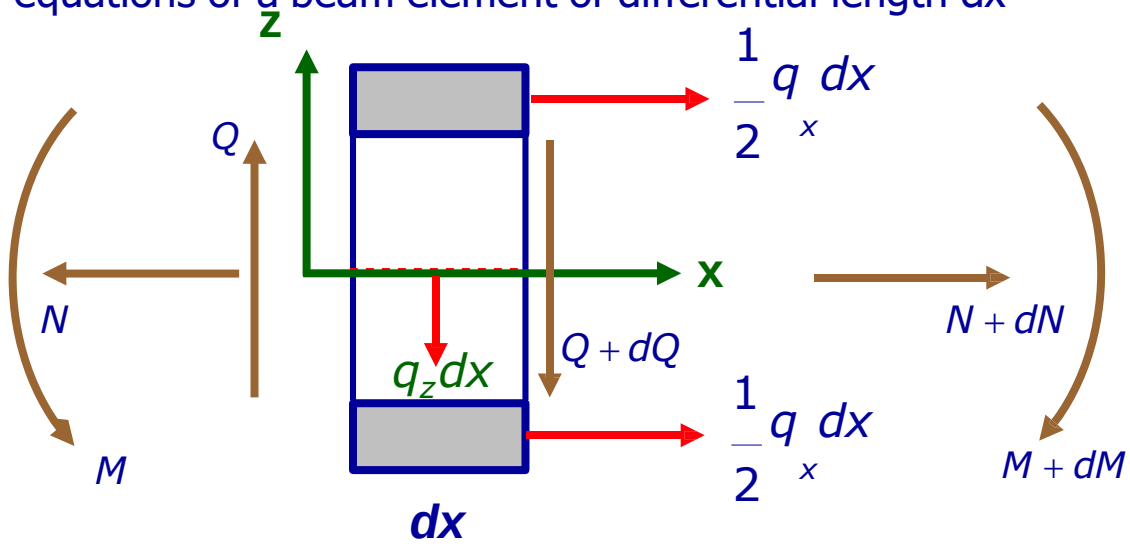
$$\frac{dQ}{dx}=-q_Z$$

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# Stress field in sandwich beams

## Shear force

Equilibrium equations of a beam element of differential length dx




- Moments equilibrium

$$-M + (M + dM) + (Q + dQ)dx + q_z dx \frac{dz}{2} = 0$$

$$dM + Qdx = 0$$

$$\frac{dM}{dx} = -Q$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Stress field in sandwich beams

## Shear force

Shear stress as a function of shear force:

$$\tau_c = -\frac{Q}{w\left(t_1 + t_c\right)}$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Strains in sandwich beams

## Chapter 4. Composite beams and plates

## Session 10. Sandwich structures

1. Introduction
2. Hypotheses
3. Stress field in sandwich beams
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5. Displacements
6. References

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# Strains in sandwich beams

## Normal stress in the face-sheets mid-plane:

$$\sigma_x(z) \approx E(z)\cdot\left(\varepsilon_x^o + \frac{z}{R}\right)$$

$$\begin{cases} z = \dfrac{t_f + t_c}{2} & \sigma_f^+ = E_f\left(\varepsilon_x^o + \dfrac{1}{R}\dfrac{t_f + t_c}{2}\right) \\ z = -\dfrac{t_f + t_c}{2} & \sigma_f^- = E_1\left(\varepsilon_x^o - \dfrac{1}{R}\dfrac{t_f + t_c}{2}\right) \end{cases}$$

Replacing the values of $\sigma_1^+$ *and* $\sigma_1^-$

$$\frac{1}{2}\left(\frac{N}{t_1 w} + \frac{2M}{t_f w(t_1 + t_c)}\right) = E_f\left(\varepsilon_x^o + \frac{1}{R}\frac{t_f + t_c}{2}\right)$$

$$\frac{1}{2}\left(\frac{N}{t_1 w} - \frac{2M}{t_f w(t_f + t_c)}\right) = E_1\left(\varepsilon_x^o - \frac{1}{R}\frac{t_f + t_c}{2}\right)$$

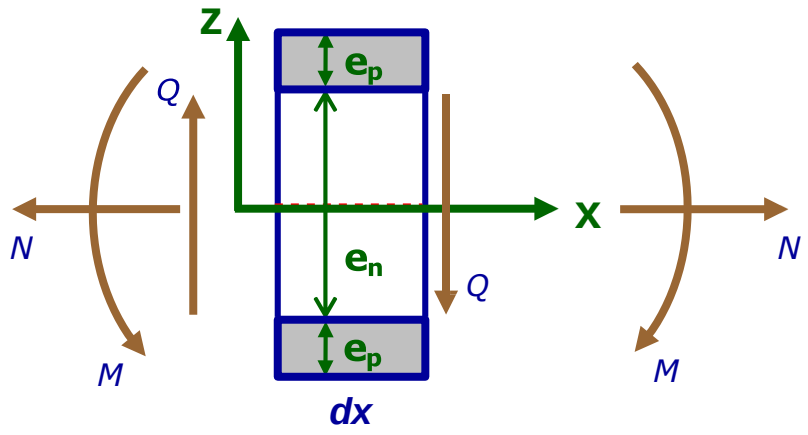
$$\varepsilon_x^o = \frac{N}{2E_f w t_f}$$

$$\frac{1}{R} = \frac{2M}{E_f w t_f (t_f + t_c)^2}$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Strains in sandwich beams

## Summary




w = beam width

tf = face-sheet thickness

$t_c$ = core thickness

N = Axial force

Q = Shear force

M = Bending moment

**Stresses:**

$$\sigma_f = \frac{1}{2}\left(\frac{N}{t_1 w} + \frac{2M}{t_1 w(t_1 + t_c)}\right)$$

$$\sigma_p = \frac{1}{2}\left(\frac{N}{t_1 w} - \frac{2M}{t_1 w(t_1 + t_c)}\right)$$

$$\tau_c = -\frac{Q}{B(t_f + t_c)}$$

**Strains:**

$$\varepsilon_x(z) = \varepsilon_x^o + \frac{N}{R} z$$

$$\varepsilon_x^o = \frac{2E_f wt_f}{}$$

$$\frac{1}{R} = \frac{2M}{E_f wt_f (t_f + t_f)^2}$$

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# Chapter 4. Composite beams and plates

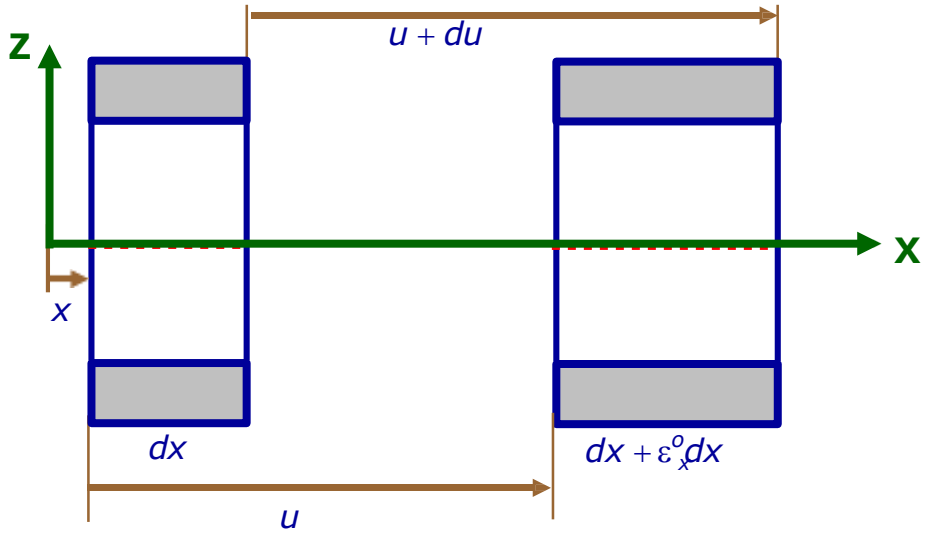
# Sandwich structures

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4. Strains in sandwich beams
5. Displacements
6. References

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

# Displacements

## Longitudinal displacements:




$$dx + (u + du) = u + dx + \varepsilon^{o}_{x} dx$$

$$\frac{du}{dx} = \varepsilon^{o}_{x}$$

Integrating from the initial section to a generic section:

$$u = u_A + \int_{x_A}^{x} \varepsilon^{o}_{x} dx$$

Substituting the value of $\varepsilon^{o}_{x}$

$$u = u_A + \int_{x_A}^{x} \frac{N}{2E_f wt_f} dx \rightarrow$$

$$u = u_A + C_n \int_{x_A}^{x} N \, dx$$

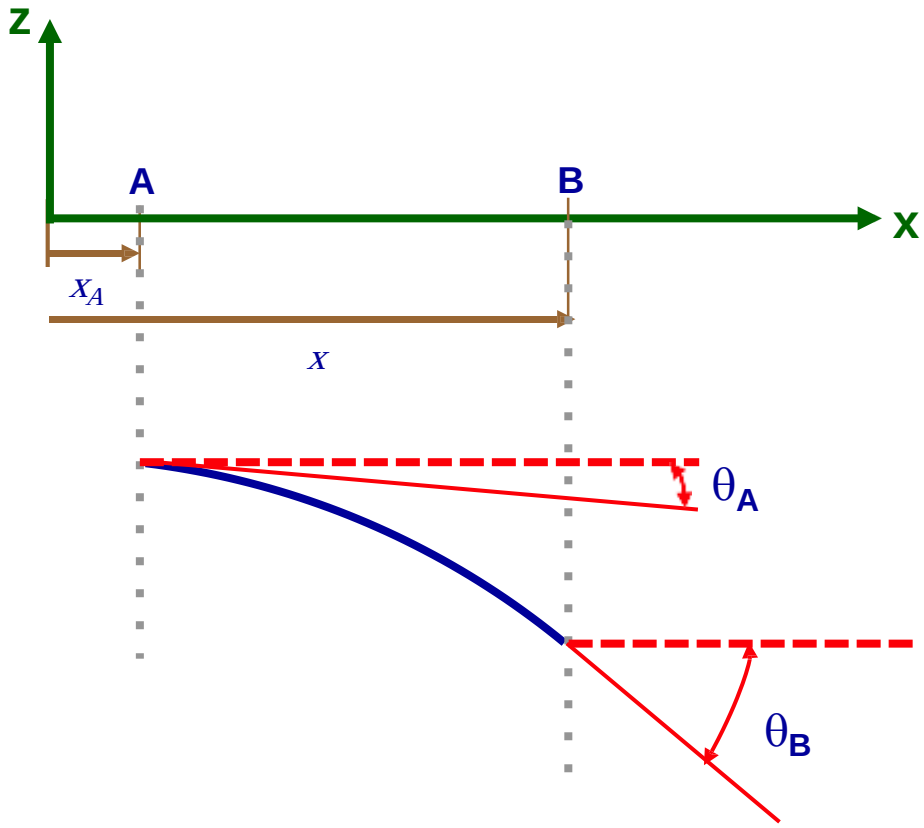
$$C_n = \frac{1}{E_f wt_f}$$

$$\langle EA \rangle = 2 \cdot E_f \, wt_f$$

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# Displacements

## Rotation




$$R \cdot d\theta = dx \left(1+\varepsilon^o_x\right)$$

$$\frac{d\theta}{dx} = \frac{1+\varepsilon^o_x}{R} \approx \frac{1}{R}$$

$$\theta - \theta_A = \int_{x_A}^{x} \frac{1}{R}\, dx$$

Substituting R by its value:

$$\theta = \theta_A + C_m \int_{x_A}^{x} M\, dx$$

$$C_m = \frac{2}{E_f w t_f \left(t_1 + t_c\right)^2}$$

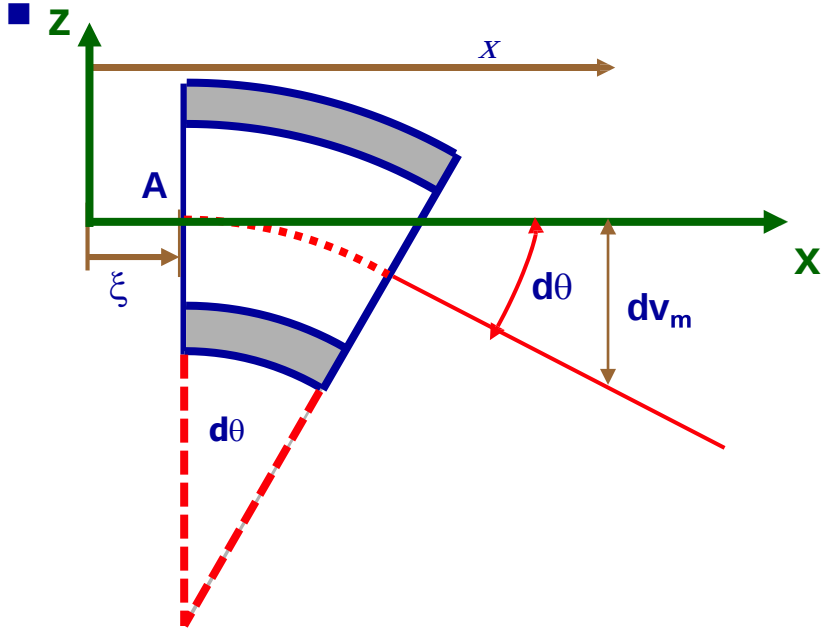
$$\langle EI \rangle = \frac{1}{2} \cdot E_f \cdot w \cdot t_f \cdot \left(t_1 + t_c\right)^2$$

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# Displacements

## Transverse displacements

- Due to rotations




$dv_m$ = section transverse displacement at x due to the relative rotation, dθ, between section ξ y other section infinitely near

$$d\theta = \frac{dv_m}{(x-\xi)}$$

$$d\theta = MC_m dx \quad \rightarrow \quad dv_m = (x-\xi)MC_m dx$$

The relative displacement between A and x can be found integrating the transverse displacements produced by the rotation of the sections between A and x. It is necessary to add the displacement due to the rotation of section A, $\theta_A$

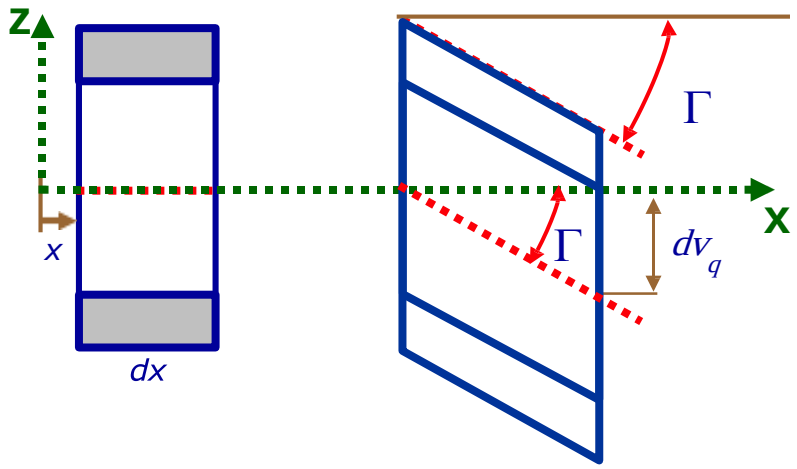
$$v_m = v_A + \theta_A(x - x_A) + C^m \int_{\xi=x_A}^{\xi=x} M(x-\xi)d\xi$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Displacements

## Transverse displacements

- Due to shear strains




There are relative transverse displacements in a section produced by shear strains

$$dv_q = \Gamma\, dx \quad \rightarrow \quad v_q = v_q^A + \int_A^x \Gamma\, dx$$

Shear strains are associated to shear stresses.

$$\gamma = \frac{\tau}{G}$$

$G$ is a material property, thus it has different values in face-sheets and core, moreover $\tau$ is not constant through face-sheet thickness. $\Gamma$ can not be identify with a shear strain $\gamma$ because this is not constant.

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# Displacements

## Transverse displacements

- Due to shear strains

Using the principle of virtual works in a section subjected to internal forces N, Q, M.

$$M \cdot d\theta + N \cdot du + Q \cdot \Gamma \, dx = \int_{Area} (\sigma_x \varepsilon_x + \tau_{xz} \gamma_{xz}) dA \cdot dx = \int_{Area} \left( \frac{\sigma_x^2}{E} + \frac{\tau_{xz}^2}{G} \right) dA \cdot dx$$

$\sigma_x$ and $\tau_{xy}$ can be replaced by their values

$$\sigma_x \begin{cases} \text{In face-sheets} & \sigma_x = \sigma_f^+ \text{ or } \sigma_f^- \\ \text{In core} & \sigma_x = 0 \end{cases}$$

$$\tau_{xy} \begin{cases} \text{In face-sheets} & \begin{cases} z = \pm (t_1 + t_c/2) \rightarrow \tau_{xy} = 0 \\ z = \pm (t_c/2) \rightarrow \tau_{xy} = \tau_n \end{cases} \\ \text{In core} & \tau_{xy} = \tau_n \end{cases}$$

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# Displacements

## Transverse displacements

- Due to shear strains

$$M\cdot\frac{d\theta}{dx}+N\cdot\frac{du}{dx}+Q\cdot\Gamma=\frac{\sigma_f^{+2}}{E_f}wt_1+\frac{\sigma_f^{-2}}{E_f}wt_1+\frac{\tau_c^2}{G_c}wt_c+\frac{2\tau_c^2}{3G_f}wt_f$$

$$\frac{du}{dx}=C_n\cdot N \qquad \frac{d\theta}{dx}=C_m\cdot M$$

$$M^2\cdot C_m+N^2\cdot C_n+Q\cdot\Gamma=M^2\cdot C_m+N^2\cdot C_n+Q^2\frac{t_c}{w(t_c+t_f)^2G_c}\cdot\left(1+\frac{2t_fG_c}{3t_cG_f}\right)$$

$$\Gamma=C_q\cdot Q$$

$$C_q=\frac{t_c}{w(t_c+t_f)^2G_c}\cdot\left(1+\frac{2t_fG_c}{3t_cG_f}\right)$$

$$\langle GA_s\rangle=\frac{w(t_c+t_1)^2G_c}{t_c\cdot\left(1+\frac{2t_fG_c}{3t_cG_f}\right)}$$

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# Displacements

## Transverse displacements

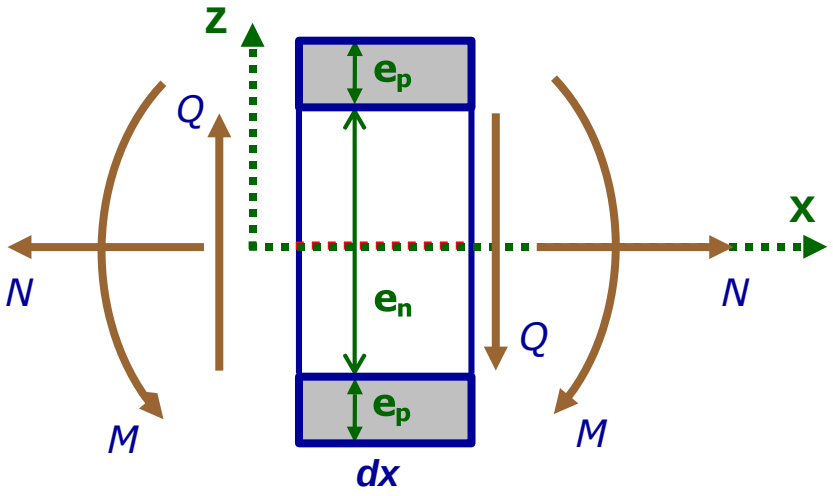
Thus global transverse displacements yields:

$$v = v_A + \theta_A (x - x_A) + C_m \int_{\xi=x_A}^{\xi=x} M(x-\xi)\, d\xi + C \int_{x_A}^{x} Q\, d\xi$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Displacements

## Summary




$$u = u_A + C_n \int_{x_A}^{x} N\, dx$$

$$\theta = \theta_A + C_m \int_{x_A}^{x} M\, dx$$

$$v = v_A + \theta_A (x - x_A) + C_m \int_{\xi = x_A}^{\xi = x} M(x - \xi)\, d\xi + C_q \int_{x_A}^{x} Q\, d\xi$$

$$C_n = \frac{1}{2 E_f w t_f}$$

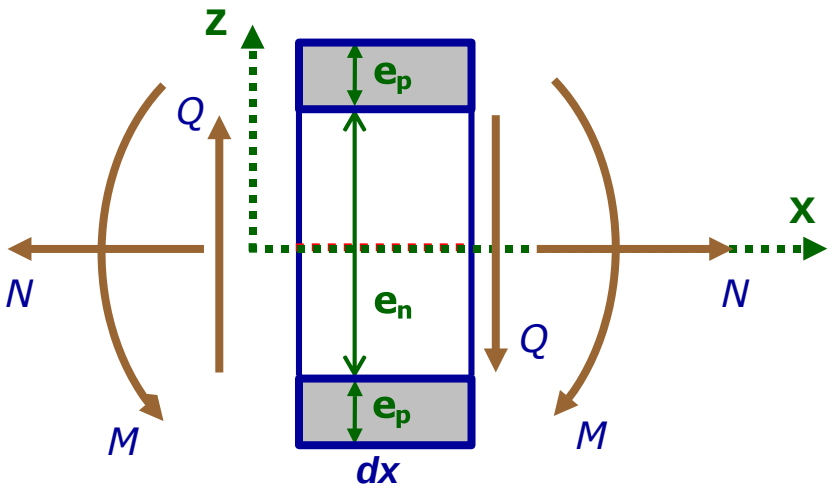
$$C_m = \frac{2}{E_f w t_f (t_1 + t_c)^2}$$

$$C_q = \frac{t_c}{w (t_c + t_f)^2 G_c} \cdot \left( 1 + \frac{2 t_f G_c}{3 t_c G_f} \right)$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# Displacements

## Summary




$$u = u_A + \int_{x_A}^{x} \frac{N}{\langle EA \rangle} dx$$

$$\theta = \theta_A + \int_{x_A}^{x} \frac{M}{\langle EI \rangle} dx$$

$$v = v_A + \theta_A (x - x_A) + \int_{\xi=x_A}^{\xi=x} \frac{M}{\langle EI \rangle} (x - \xi) d\xi + \int_{x_A}^{x} \frac{Q}{\langle GA_c \rangle} d\xi$$

$$\langle EA \rangle = 2 E_f w t_f$$

$$\langle EI \rangle = \frac{1}{2} E_f w t_f \left( t_f + t_c \right)^2$$

$$\langle GA_s \rangle = \frac{w \left( t_c + t_f \right)^2 G_c}{t_c \left( 1 + \frac{2 t_f G_c}{3 t_c G_f} \right)}$$

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# References

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6. **References**

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# References

**The Behaviour of Structures Composed of Composite Materials**

J.R. Vinson y R.L. Sierakowski

Kluwer Academic Publishers. 2002.

Cap. 4: Beams, columns and Rods of Composite Materials

**Mechanics of laminated composite plates. Theory and analysis**

J.N. Reddy

CRC Press. 1997.

Cap 6: One- Dimensional Analysis of Laminated Plates