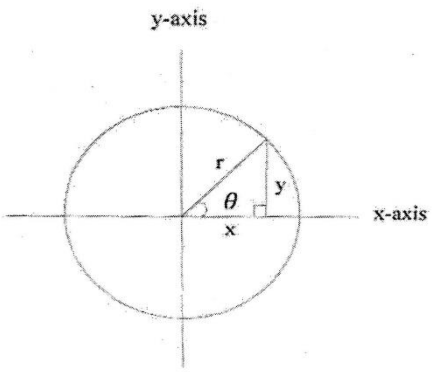
1-1 Trigonometric Functions: الدوال المثلثية

Given central circle with a radius (r), The point P(x,y) could be any point on the circle then:

- Sine of $\theta \to \sin\theta = \frac{y}{r}$
- Cosine of $\theta \to \cos\theta = \frac{x}{r}$
- Tangent of $\theta \to \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$
- Secant of $\theta \to \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$
- Cosecant of $\theta \to \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$
- Cotangent of $\theta \to \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$



From above triangle we can say:

$r^2 = x^2 + y^2$ (فيثاغورس)

But: $x = r\cos\theta$ & $y = \sin\theta$

$$\therefore r^2 = (r\cos\theta)^2 + (r\sin\theta)^2$$

$$r^2 = r^2cos^2\theta + r^2sin^2\theta$$

$$r^2 = r^2(cos^2\theta + sin^2\theta)$$

$$sin^2\theta + cos^2\theta = 1$$

Some trigonometric Identities:

- $sin^2x + cos^2x = 1$ ……………………(1)
- $tan^2x + 1 = sec^2x$ (ناتجة من قسمة المعادلة رقم (1) على cos^2x)
- $1 + cot^2x = csc^2x$ (ناتجة من قسمة المعادلة رقم (1) على sin^2x)
- $\sin 2x = 2\sin x\cos x$
- $\cos 2x = cos^2x - sin^2x$
- $\cos 2x = (1 - sin^2x) - sin^2x \longrightarrow \cos 2x = 1 - 2sin^2x$

  $2sin^2x = 1 - \cos 2x \to \therefore\ sin^2x = \frac{1}{2}(1 - \cos 2x)$

  Also, $cos^2x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(x + y) = \sin x\cos y + \cos x\, siny$
- $\sin(x - y) = \sin x\cos y - \cos x\sin y$
- $\cos(x + y) = \cos x\cos y - \sin x\sin y$
- $\cos(x - y) = \cos x\cos y + \sin x\sin y$

- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

- $\sin A \,.\, \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \,.\, \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$
- $\sin A \,.\, \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

- **Prove that**: $sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Right hand side $= \frac{1}{2}(1 - \cos 2x)$

$$= \frac{1}{2}[1 - (cos^2 x - sin^2 x)]$$

$$= \frac{1}{2}[1 - (1 - sin^2 x - sin^2 x)$$

$$= \frac{1}{2}[1 - 1 + 2sin^2 x]$$

$sin^2 x =$ the left side $\Rightarrow$ ∴ o.K.

قياس بعض الزوايا الخاصة

+ (sin) الربع الثاني	+ (sin, cos, tan…) الربع الأول
+ (tan) الربع الثالث	+ (cos) الربع الرابع

❖ اضافة أو طرح أي عدد من الدورات لا يغير من قيمة الزاوية (2π دورة واحدة & 4π دورتان & 6π ثلاث دورات و هكذا ......).

❖ $360 = 2\pi$ & $270 = \frac{3\pi}{2}$ & $180 = \pi$ & $90 = \frac{\pi}{2}$

- الزوايا المكملة (2π & π) : موقع الربع يحدد الأشارة و تبقى الدالة نفسها مثلا دالة ال sin تبقى sin كما في الامثلة ادناه :-

$\sin(\pi-\theta)=+\sin\theta$ الربع الثاني	$\cos(\pi+\theta)=-\cos\theta$ الربع الثالث
$\sin(\pi+\theta)=-\sin\theta$ الربع الثالث	$\tan(\pi-\theta)=-\tan\theta$ الربع الثاني
$\cos(\pi+\theta)=-\cos\theta$ الربع الثالث	$\tan(\pi+\theta)=+\tan\theta$ الربع الثالث

$$\left.\begin{aligned}\sin(2\pi-\theta)&=-\sin\theta\\ \tan(2\pi-\theta)&=-\tan\theta\\ \cos(2\pi-\theta)&=+\cos\theta\end{aligned}\right\}$$ زوايا في الربع الرابع

لاحظ انه:-

Sin(-θ) =- sin θ , cos (-θ) = cosθ , tan (-θ) = -tan (θ)

- الزوايا المتممة $(\frac{3\pi}{2}\&\frac{\pi}{2})$ :: موقع الربع يحدد الاشارة و تقلب الدالة الى متتمتها. مثلا دالة ال sin في الربع الأول تكون موجبة ثم تقلب الى cos .

زوايا في مختلف الارباع	
$\sin(\frac{\pi}{2}+\theta)=+\cos\theta$ الربع الثاني	$\cos(\frac{\pi}{2}+\theta)=-\sin\theta$ الربع الثاني
$\cos(\frac{\pi}{2}+\theta)=-\ \sin\theta$ الربع الثاني	$\tan(\frac{\pi}{2}+\theta)=-cot\theta$ الربع الثاني
$\sin\left(\frac{\pi}{2}-\theta\right)=+\cos\theta$ الربع الاول	$\tan(\frac{\pi}{2}-\theta)=+\cot\theta$ الربع الاول
$\sin(\frac{3\pi}{2}-\theta)=-\cos\theta$ الربع الثالث	$\sin(\frac{3\pi}{2}+\theta)=-\cos\theta$ الربع الرابع
$\cos(\frac{3\pi}{2}-\theta)=-\sin\theta$ الربع الثالث	$\tan(\frac{3\pi}{2}-\theta)=+\cot\theta$ الربع الثالث
$\cos(\frac{3\pi}{2}+\theta)=+\sin\theta$ الربع الرابع	$\tan(\frac{3\pi}{2}+\theta)=-\cot\theta$ الربع الرابع

Some examples about Trigonometric Functions

- $sin\ 10x=2\sin 5x\ cos5x$
- $cos4x\ \ =cos^2 2x-sin^2 sx$
- $cos^2 7x\ =\frac{1}{2}(1+\cos 14x)$
- $sin^2 15=\frac{1}{2}(1-\cos 30)$
- $cos\ 330=cos\ (270+\ 60)=+\sin 60$

  or $cos\ 330=cos\ (360-30)=cos30$
- $sin\ 150=sin(180-30)=\ +sin\ 30$

1-2 Functions: الدوال

1-2-1 the Domains and the Ranges:

Domain: هو مجال القيم الحقيقة ل (x) التي تأخذ قيم حقيقة ل (y).

و صيغة المعادلة تكون: Equation $\rightarrow y = f(x)$

Range: هو مجال القيم الحقيقة ل(y) التي تأخذ قيم حقيقة ل(x).

و صيغة المعادلة تكون: Equation $\rightarrow x = f(y)$

Domain and Range are divided in to bounded and Infinite Intervals:

Bounded Intervals الفترات المحددة		Infinite Intervals الفترات الغير المحددة	
1	$a < x < b$ a(———)b → open intervals D_f, $:a < x < b$ or (a, b)	1	$-\infty < x < \infty$ $-\infty \leftarrow 0 \rightarrow \infty$ D_f : $-\infty < x < \infty$ or $(-\infty, \infty)$ Or R (all real number)
2	$a \leq x \leq b$ $a[$———$]b$ $\rightarrow$ $close$ $interval$ D_f, $:a \leq x \leq b$ or $[a, b]$	2	$a < x$ $(a \rightarrow \infty$ D_f $:x > a$ or (a, ∞)
3	$a \leq x < b$ A[———)b → half-open interval D_f, $:a \leq x < b$ or $[a, b)$	3	$a \leq x$ [a→∞ D_f $:x \geq a$ or $[a, \infty)$
4	$a < x \leq b$ $a($———$]b$→ half-open interval D_f, R_f $:a < x \leq b$ or $(a, b]$	4	$x < b$ $-\infty \leftarrow b)$ D_f $:x < b$ or $(-\infty, b)$
		5	$x \leq b$ $-\infty \leftarrow b]$ D_f, $:x \leq b$ or $(-\infty, b]$

<u>**Example 1:**</u> $\boldsymbol{y = f(x) = x + 15}$

D_f: R

To find the range:- Re- write the function as x=f(y)

$$y = x + 15 \rightarrow x = f(y) = y - 15$$

∴ D_f : R

Example 2:

$$\boldsymbol{y = \frac{3}{x-2}}$$ المقام $\boldsymbol{\neq 0}$

الشرط $\rightarrow x-2 \neq 0 \rightarrow x \neq 2 \quad \rightarrow \quad \therefore$ **D_f : R ∕ {2}**

To find the range: - Re- write the function as x=f(y)

$$y = \frac{3}{x-2} \rightarrow yx - 2y = 3$$

$$yx = 3 + 2y \rightarrow x = f(y) = \frac{3+2y}{y}$$

∴ **R_f : R ∕ {0}**

Example 3:

$\boldsymbol{y = \sqrt{x-1}}$ لا يجوز أن تكون القيمة سالبة تحت الجذر (قيمة خيالية)

الشرط$\rightarrow x-1 \geq 0 \rightarrow x \geq 1$

D_f : x: x ≥ $\boldsymbol{1\ or\ [1,\infty)}$

To find the range:- Re- write the function as x=f(y)

$$y = \sqrt{x-1} \rightarrow y^2 = x-1$$

$$x = y^2 + 1$$

R_f: y: y ≥ $\boldsymbol{0\ or\ [0,\infty)}$ نأخذ القيم الموجبة و الصفر فقط لأن الدالة الأصلية هي دالة جذرية

Example 4:

$$\boldsymbol{y = \frac{1}{\sqrt{x-1}}}$$ هنا الجذر موجود في مقام الدالة

الشرط $\rightarrow x-1 > 0 \rightarrow x > 1$

D_f: x: x $> 1\ or\ (1,\infty)$

To find the range:- Re- write the function as x=f(y)

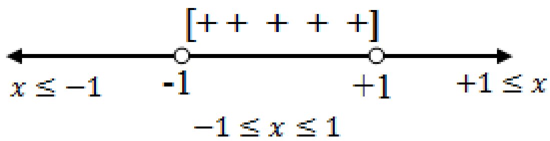
$$y = \frac{1}{\sqrt{x-1}} \rightarrow y^2 = \frac{1}{x-1}$$

$$y^2x - y^2 = 1 \rightarrow x = \frac{1-y^2}{y^2} \rightarrow x = 1 + \frac{1}{y^2}$$

$\therefore$ R_f: y:y> 0 نأخذ القيمة الموجبة فقط لأن الدالة الأصلية هي دالة جذرية و كسرية

Example 5:

$$\boldsymbol{y = \sqrt{1-x^2}}$$



الشرط→ $1 - x^2 \ge 0$

$(1 - x)(1 + x) \ge 0$

من رسم الدالة واختبار الفترات ⟶ $\therefore$ D_f : $\boldsymbol{-1 \le x \le 1 \; or \; [-1, 1]}$

To find the range:- Re- write the function as x=f(y)

$y = \sqrt{1 - x^2} \to y^2 = 1 - x^2$

$x^2 = 1 - y^2 \to x = \sqrt{1 - y^2}$

الشرط → $1 - y^2 \ge 0$

$(1 - y)(1 + y) \ge 0$

من رسم الدالة واختبار الفترات ⟶ $\therefore$ R_f: $\boldsymbol{0 \le y \le 1 \; or \; [0, 1]}$

Example 6:

$\boldsymbol{y = x + \frac{1}{x}}$ **x ≠ 0**

D_f: R / {0} مجال الدالة الكلي هو حاصل تقاطع مجال الدالتين

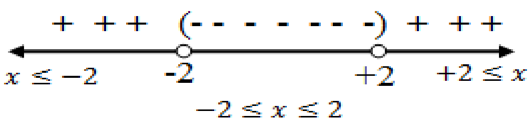
To find the Range: $y = x + \frac{1}{x} \to y = \frac{x^2+1}{x} \to yx = x^2 + 1$

المعادلة العامة ⟶ $ax^2 + bx + c = 0$
قانون الدستور $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x^2 - yx + 1 = 0$

$a = 1 \;\&\; b = -y \;\&\; c = 1$

$x = \frac{y \pm \sqrt{y^2-(4*1*1)}}{2*1}$



الشرط → $y^2 - 4 \ge 0$

$(y - 2)(y + 2) \ge 0$

من رسم الدالة R_f: $\boldsymbol{y \ge 2 \cup y \le -2 \; or \; R/(-2, 2)}$

Example 7:

$$\boldsymbol{y = \sqrt{x^2 - 3x}}$$

الشرط $\longrightarrow x^2 - 3x \geq 0$ $\qquad$ $\underline{+ + + +\ 0[- - - -]3 + + + +}$

$x(x-3) \geq 0$ $\qquad$ $\leftarrow \qquad \rightarrow$

من رسم الدالة $\longrightarrow$ **D$_f$:** $\boldsymbol{x \geq 3 \cup x \leq 0\ or\ R\ /\ (0,3)}$

To find the Range:

$y = \sqrt{x^2 - 3x} \rightarrow y^2 = x^2 - 3x$

$x^2 - 3x - y^2 = 0$

$a = 1\ \&\ b = -3\ \&\ c = -y^2$ حل المعادلة من الدرجة الثانية بطريقة الدستور

$x = \frac{3 \pm \sqrt{9 - [4*1*(-y^2)]}}{2*1} \longrightarrow \quad 9 + 3y^2 \geq 0$

R$_f$: $\boldsymbol{y \geq 0}$ **or[0,∞)** اصل الدالة عي دالة جذرية ولذا يتم استثناء القيم السالبة من المجال المقابل

Example 8:

Find the domain only for $\boldsymbol{y = \sqrt{x-3} + \sqrt{3-2x}}$

$\sqrt{x-3} \quad + \quad \sqrt{3-2x}$

$\downarrow \qquad\qquad \downarrow$

$Df1 \Rightarrow \boldsymbol{x - 3 \geq 0} \ \longrightarrow \boldsymbol{x \geq 3}$

$Df2 \Rightarrow \ \boldsymbol{3 - 2x \geq 0 \Rightarrow \ 3 \geq 2x \longrightarrow \frac{3}{2} \geq x}$ $\qquad$ $\underline{+++3/2]-------[3+++}$ $\quad \leftarrow \qquad \rightarrow$

من رسم مجال الدوال واختبار الفترات $\therefore$ **D$_f$ = ∅** **كمية خالية** $\qquad (Df1) \cap (Df2) = \emptyset$

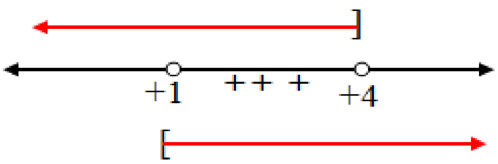
Example 9: **Find the domain only for**

$\boldsymbol{y = \sqrt{4-x} + \sqrt{x-1}}$

$= \sqrt{4-x} \qquad \sqrt{x-1}$

$\downarrow \qquad\qquad \downarrow$

$4 - x \geq 0 \qquad x - 1 \geq 0$



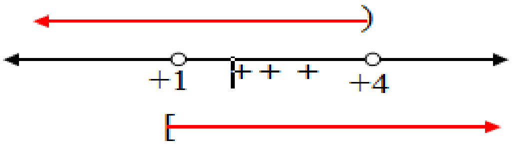
D$_f$:- $\boldsymbol{1 \leq x \leq 4\ or}$ **[1,4]**

$4 \geq x \qquad\qquad x \geq 1$

من رسم مجال الدوال واختبار الفترات ∴ **D_f :- $1 \leq x \leq 4$ *or* [1,4]**

Example 10:

Find the domain only for:-



$$y = \sqrt{\frac{x-1}{4-x}}$$

$$y = \frac{\sqrt{x-1}}{\sqrt{4-x}}$$

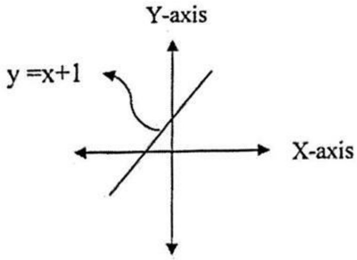
$Df1 \Rightarrow x - 1 \geq 0 \;\Rightarrow x \geq 1$

$Df2 \Rightarrow 4 - x > 0 \Rightarrow 4 > x$

∴ **D_f :- $1 \leq x < 4$ *or* [1,4)**

Note: the domain and the range of any function could be found also by graphing the function itself.

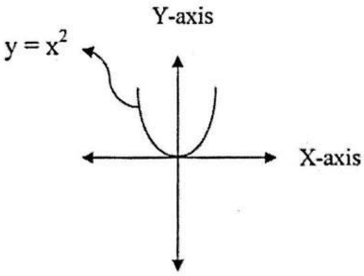
The projection of the graph of a function (f) on the x-axis is the domain (D_f) and on the y-axis is the range (R_f).



Example 11: $y = x + 1$

D_f: R

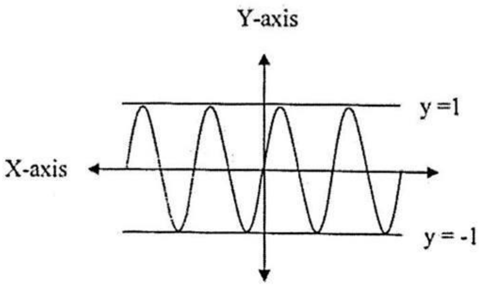
R_f: R



Example 12: $y = x^2$

D_f: R

R_f: R



Example 13: $y = \sin x$

D_F: R

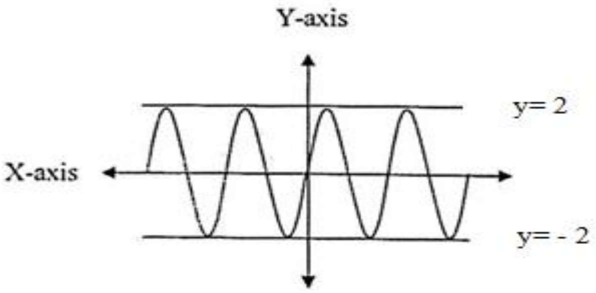
R_F = [-1,1]

Example 14:

$$y = 2\sin x$$

D_f: R

R_f = [-2, 2]

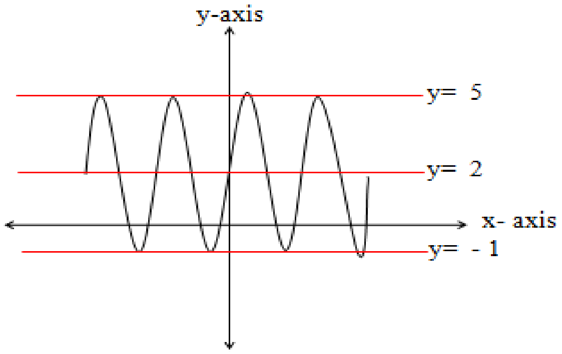


Example 15:

$$y = 2 + 3\sin x$$

D_f: R

R_f = [-1, 5]

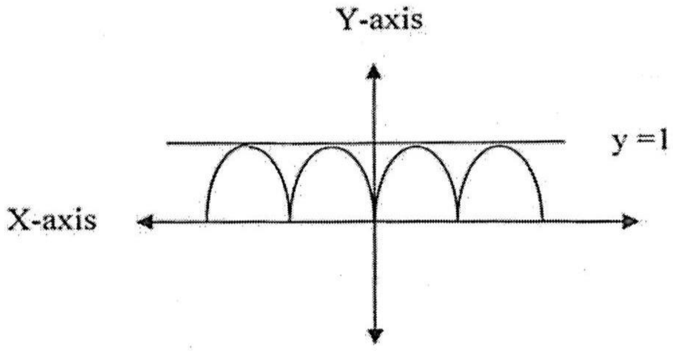


Example 16:

$$y = sin^2 x$$

D_f: R

R_f = [0,1]

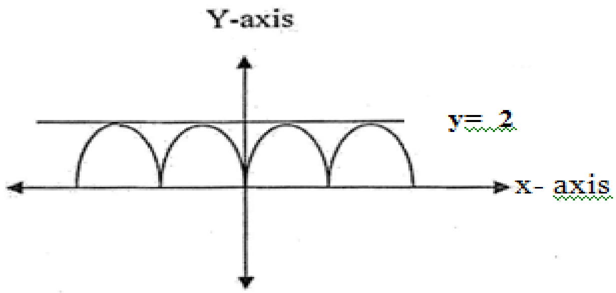


Example 17:

$$y = -2sin^2 x$$

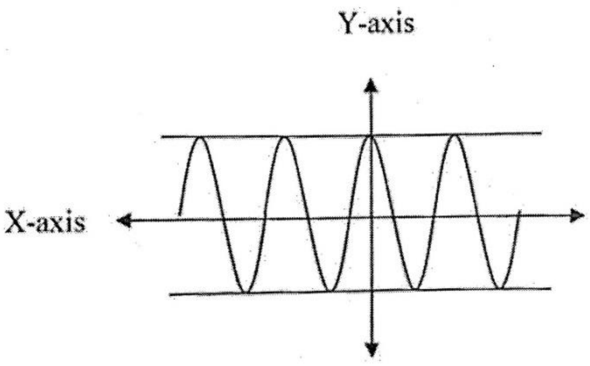
D_f: R

R_f = [-2,0]



Example 18:

$$y = \cos x$$



D_f: R

R_f = [-1,1]

1-2-2 Even and odd Functions: الدوال الزوجية و الدوال الفردية

A function (f) is called:

- Even if → f(-x) = + f(x)
- Odd if → f(-x) –f(x)

Examples:

1) $\boldsymbol{y = x^2}$

$f(-x) = (-x)^2 = x^2 = +f(x) \rightarrow \therefore\ even\ function$

2) :$\boldsymbol{y = x^3}$

$f(-x) = (-x)^3 = -x^3 = -f(x) \rightarrow \therefore\ odd\ function$

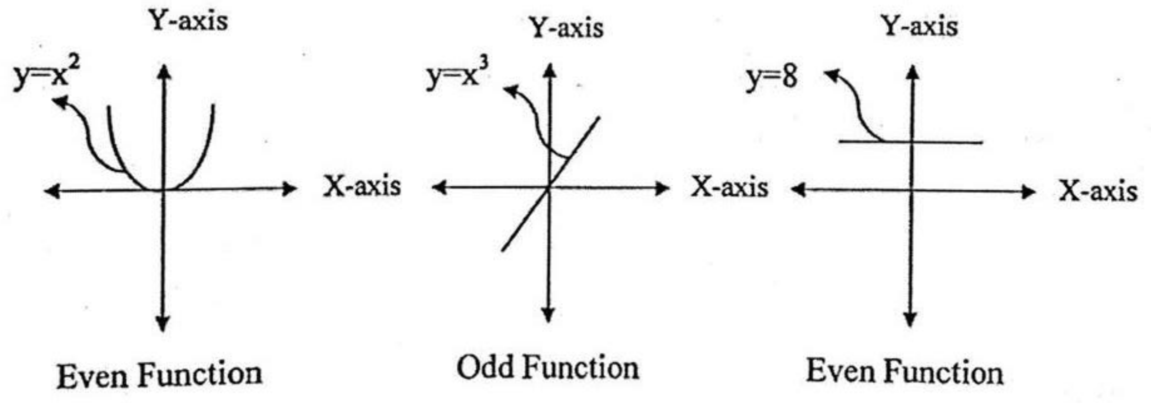
3) $\boldsymbol{y = 8}$

$f(-x) = 8\ \rightarrow \therefore\ even\ function$

Note (1) : For all x ∈ D_f:

- Even if → the function is symmetric about the y-axis.
- Odd if → the function is symmetric about the origin.

Examples:



Note (2) :

- $Odd \pm Odd = Odd\ \&\ Even \pm Even = Even$
- $Odd * Odd = Even \& Even * Even = Even$
- $Odd/Even = Odd = Even/Odd$
- $Odd * Even = Odd = Even * Odd$

Examples

1) $y = sin\, x$

Odd function (symmetric about the origin)

2) $y = cos\, x$

Even function (symmetric about the y-axis)

3) $y = tan\, x$

$$y = \frac{\sin x}{\cos x} = \frac{Odd}{Even} = Odd\ function$$

4) $\mathbf{f(x)} = \frac{x^2+x^4}{x+\sin x}$

$$f(x) = \frac{even + Even}{Odd + Odd} = \frac{Even}{Odd} = Odd\ function$$

5) $\mathbf{f(x)} = x^3 - 2$

$f(x) = Odd - Even = neither\ Even\ nor\ Odd$

6) $\mathbf{f(x)} = \frac{x^3+x^5}{sinx+2}$

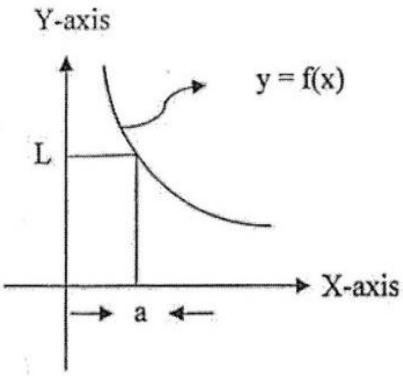
$$f(x) = \frac{Odd + Odd}{Odd + Even} = neither\ Even\ nor\ Odd$$

1-2-3 Limits of a function: الغايات

Definition

$\lim_{x\to a} f(x) = L$ Mean that when a value of (x)

Close to (a) ⇒f(x) approaches the limiting value (L).



$\lim_{x\to a+} f(x) = L$ Mean that (x) approaches (a) from th

$\lim_{x\to a-} f(x) = L$ Mean that (x) approaches (a) from th

Note:

If $\lim_{x\to a+} f(x) = L = \lim_{x\to a^-} f(x) = L$ we say that $\lim_{x\to a} f(x) = L$ exist,

Otherwise the limit doesn't exist.

Example 1:

Find $\lim_{x\to 1} f(x)$ when $f(x) = \begin{bmatrix} x^2+1 & \text{when } x \geq 1 \\ 3 & \text{when } x < 1 \end{bmatrix}$

$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} (x^2+1) = (1)^2 + 1 = 2$

$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} (3-x) = 3 - 1 = 2$

$\because \lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x) = 2 \to \therefore \lim_{x\to 1} f(x)$ Exist

Example 2:

Find $\lim_{x\to 0} f(x)$ where $f(x)$ $= \begin{bmatrix} x^2+1 & \text{when } x \geq 0 \\ x & \text{when } x < 0 \end{bmatrix}$

$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} (x^2+1) = 0 + 1 = 1$

$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} (x) = 0$

$\because \lim_{x\to 0+} f(x) \neq \lim_{x\to 0-} f(x) \to \therefore \lim_{x\to 0} f(x)$ Doesn't exist

Properties of Limits: خصائص الغايات

Let: $\lim_{x\to a} f(x) = L1$

$\lim_{x\to a} g(x) = L2$

K is a constant, then:

1) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L1 \pm L2$
2) $\lim_{x\to a}[f(x) * g(x)] = \lim_{x\to a} f(x) * \lim_{x\to a} g(x)$
3) $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ but $\lim_{x\to a} g(x) \neq 0$
4) $\lim_{x\to a} K * f(x) = K * \lim_{x\to a} f(x)$
5) $\lim_{x\to a} K = K$

6) $\lim_{x\to a} x = a$

7) $\lim_{x\to\infty} \frac{1}{x} = 0 \ \ and \ \ \lim_{x\to\infty} \sqrt{x} = \infty$

8) $\lim_{x\to 0+} \frac{1}{x} = +\infty \ and \ \ \lim_{x\to 0-} \frac{1}{x} = -\infty$ but $\lim_{x\to 0} \frac{x}{1} = 0$

9) $\lim_{x\to 0} \sin x = 0$; $\lim_{x\to 0} \cos x = 1 \ and \ \lim_{x\to 0} \tan x = 0$

10) $\lim_{x\to 0} \frac{\sin x}{x} = 1 \ and \ \lim_{x\to 0} \frac{x}{\sin x} = 1$

11) $\lim_{x\to 0} \frac{\tan x}{x} = 1 \ and \ \lim_{x\to 0} \frac{x}{\tan x} = 1$

12) $\lim_{x\to 0} \frac{1-cos\ x}{x} = 0$

13) $\lim_{x\to a} \sin\left(\frac{x^2}{\pi+x}\right) = \sin\left(\lim_{x\to a} \frac{x^2}{\pi+x}\right)$

Note: sin or cos or any trigonometric function is the same

14) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} \ \ and \ \lim_{x\to a} \frac{1}{x^n} = \left[\lim_{x\to a} \frac{1}{x}\right]^n$

<u>Note:</u>
Undefined expression in limits:
$\frac{0}{0}, \frac{\infty}{\infty}, \frac{0}{\infty}, \frac{\infty}{0}, 0 * \infty, \infty * \infty, \infty - \infty$

$but\ we\ can\ say\ \infty + \infty = \infty$

أساليب الحل الممكن اتباعها في حل أسئلة الغايات:

1) التعويض المباشر اذا كان الناتج معرف.
2) باستخدام طرق التحليل المختلفة أو الضرب بالمرافق اذا كانت ليست دوال مثلثية.
3) باستخدام الخصائص من 9 الى 13 اذا كانت دوال مثليثة.
4) اذا كانت $x \to \infty$ فهنالك ثلاث طرق للحل:

أولا: اذا كانت دوال مثلثية نحول x الى متغير أُخر و ليكن $\frac{1}{y}$ مثلا و عندما $x \to \infty$ فأن $y \to 0$

ثانيا: اذا كانت دوال كسرية نقسم على اكبر اس موجود في المقام.

ثالثا: اذا كانت دوال غير كسرية و ليست مثلثية نحولها الى دوال كسرية بالضرب في المرافق ثم نقسم على اكبر اس موجود في المقام.

5) باستخدام طريقتين أو أكثر.

Evaluate the following limits:

Examples:

1) $\lim\limits_{x\to 1}\frac{x^2+1}{x} = \frac{1+1}{1} = 2$

2) $\lim\limits_{x\to 1}\frac{x^2-1}{x+1}$

$$\lim_{x\to 1}\frac{(x-1)(x+1)}{(x+1)} = \lim_{x\to 1}(x-1) = -1-1 = -2$$

3) $\lim\limits_{x\to 1}\frac{x^3-1}{x-1}$

$$= \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{(x-1)} = \lim_{x\to 1}(x^2+x+1) = 1+1+1 = 3$$

4) $\lim\limits_{x\to 0}\frac{x}{1-\sqrt{1-x}}$

$= \lim\limits_{x\to 0}\frac{x}{1-\sqrt{1-x}} * \frac{1+\sqrt{1-x}}{1+\sqrt{1-x}}$ الضرب في مرافق المقام

$$= \lim_{x\to 0}\frac{x(1+\sqrt{1-x})}{1-(1-x)} = \lim_{x\to 0}\frac{x(1+\sqrt{1-x})}{1-1+x} = \lim_{x\to 0}\left(1+\sqrt{1-x}\right) = 1+\sqrt{1} = 2$$

5) $\lim\limits_{x\to 0}\frac{\sin 3x}{x}$

$$= \lim_{x\to 0}\frac{\sin 3x}{x} * \frac{3}{3} = 1 * 3 = 3$$

6) $\lim\limits_{x\to 0}\frac{\sin^2 3x}{x^z}$

$$= \lim_{x\to 0}\frac{\sin^2 3x}{x^2} * \frac{3*3}{3*3} = 9*1*1 = 9$$

7) $\lim\limits_{x\to 1}\frac{\sin 3x}{\sin 5x}$

$$= \lim_{x\to 1}\frac{\sin 3x * \frac{3x}{3x}}{\sin 5x * \frac{5x}{5x}} = \frac{3}{5}$$

8) $\lim\limits_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)} = \lim\limits_{x\to 0}\frac{\sin x}{x} * \lim\limits_{x\to 0}\frac{\sin x}{(1+\cos x)} = 1 * \frac{0}{2} = 0$

9) $\lim\limits_{x\to 0}\frac{1-\cos x}{x}$

$$=\lim_{x\to 0}\frac{1-\cos x}{x}*\frac{1+\cos x}{1+\cos x}=\lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}=\lim_{x\to 0}\frac{\sin x}{x}*\lim_{x\to 0}\frac{\sin x}{(1+\cos x)}=1*\frac{0}{2}=0$$

10) $\lim\limits_{x\to\pi}\frac{\sin x}{x-\pi}$

Assume $y=x-\pi\Rightarrow x=y+\pi$

$when\;\; x\to\pi\Rightarrow y\to 0$

$$=\lim_{y\to 0}\frac{\sin(y+\pi)}{y}=\lim_{y\to 0}\frac{\sin y\cos\pi+\cos y\sin\pi}{y}=\lim_{y\to 0}\frac{-\sin y}{y}=-1$$

11) $\lim\limits_{x\to 2}\frac{\cos\frac{\pi}{x}}{x-2}$

$$=\lim_{x\to 2}\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{x}\right)}{x-2}$$

Assume $y=x-2\Rightarrow x=y+2$

$when\;\; x\to 2\Rightarrow y\to 0$

$$=\lim_{y\to 0}\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{y+2}\right)}{y}=\lim_{y\to 0}\frac{\sin\left(\frac{y\pi+2\pi-2\pi}{2(y+2)}\right)}{y}$$

$$=\lim_{y\to 0}\frac{\sin\left(\frac{y\pi}{2(y+2)}\right)}{y}*\frac{\frac{\pi}{2(y+2)}}{\frac{\pi}{2(y+2)}}=\lim_{y\to 0}\frac{\pi}{2(y+2)}=\frac{\pi}{4}$$

<u>Example 11:</u> $\lim\limits_{x\to\infty} x\sin\left(\frac{2}{x}\right)$

Assume $x=\frac{1}{y}\to y=\frac{1}{x}$

$when\;\; x\to\infty\Rightarrow y\to 0$

$$=\lim_{y\to 0}\frac{1}{y}\sin 2y$$

$$=\lim_{y\to 0}\frac{\sin 2y}{y}*\frac{2}{2}=2$$

الاقتراب من الـ (∞)

Example 12: $\lim_{x\to\infty}\frac{4x^3-2x^2+1}{3x^3-5}$

$$=\lim_{x\to\infty}\frac{\frac{4x^3}{x^3}-\frac{2x^2}{x^3}+\frac{1}{x^3}}{\frac{3x^3}{x^3}-\frac{5}{x^3}}=\frac{4}{3}$$

ملاحظة: في حال كون الدالة كسرية والاقتراب من ال ∞ ، نقسم البسط والمقام على اكبر اس في المقام

Example 14: $\lim_{x\to\infty}\left(\sqrt{x^2+2x}-x\right)$

$$=\lim_{x\to\infty}(\sqrt{x^2+2x}-x)*\frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x}=\lim_{x\to\infty}\frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x}$$

$$=\lim_{x\to\infty}\frac{\frac{2x}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{2x}{x^2}}\;+\;\frac{x}{x}}=\frac{2}{2}=1$$

Continuity of a function:

Continuity of a moving particle on a single path without unbroken curve, gaps, jumps, or holes such curve can be said to be as continuous.

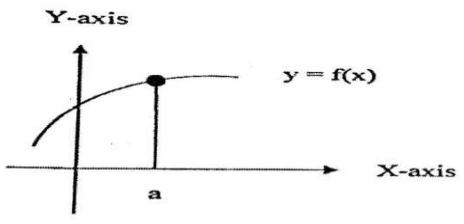


Figure (1) continuous function

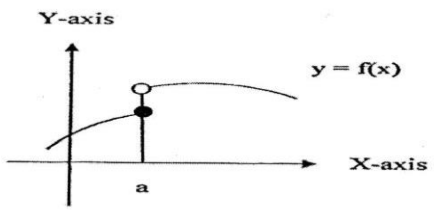


Figure (2) not continuous function

A function is said to be continuous at $x=a$ if the following are satisfied:

1) The $f(x)$ is exist or defined.
2) $\lim_{x\to a} f(x)$ exist.
3) $\lim_{x\to a} f(x)=f(a)$

Otherwise the function is not continuous.

Example 1: check if the function is continuous at $x=5, \& \; x=0$

where $f(x)=\begin{cases} x^2-1 & When\ x\geq 5 \\ x & When\ x<5 \end{cases}$

at $x = 5$

$f(5) = (5)^2 - 1 = 24$

$\lim_{x\to5+} x^2 - 1 \to \lim_{x\to5+} 25 - 1 = 24$

$\text{Lim}_{x\to5-} \to \lim_{x\to5-} 5 = 5$

$\lim_{x\to5+} f(x) \neq \lim_{x\to5-} f(x)$

The limit does not exist, therefore the function not continuous at $x = 5$

At $x = 0$

$f(0) =$ does not exist, therefore the function not continuous at $x = 0$

<u>Example 2:</u> Find the constant (a) and (b) if the function is:

$$f(x) = \begin{cases} x^2 + a & When\ x \geq 0 \\ 3 + b & When -1 \leq x < 0 \\ x + 5 & When\ x < -1 \end{cases}$$

And the function is continuous at $x = 0$ and $x = -1$

$f(0) = (0)^2 - a = a$

$\lim_{x\to0+} x^2 + a = a$

$\lim_{x\to0-} 3 + b = 3 + b$

$\lim_{x\to1-} x + 5 = 4$

The limit must be exist so $4 = 3 + b$ then $b = 1$

Then the function equal the limit value $4 = 3 + b$ then $b = 1$ substitute in equation (1) then $a = 3 + 1 = 4$

<u>Example :</u> For $x \neq 2$ the function is equal to $\frac{x^2+3x-10}{x-2}$, find the value of $f(2)$ to make the function continuous at $x = 2$.

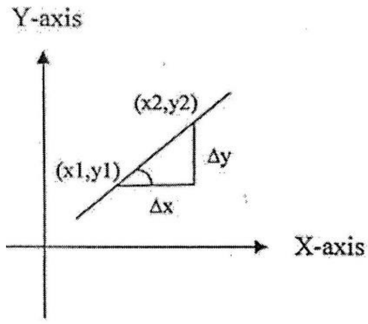
The limit to be exist must be equal from the lift to the right so:

$\lim_{x\to2} \frac{x^2+3x-10}{x-2} \to \lim_{x\to2} \frac{(x+5)(x-2)}{(x-2)}$

$\lim_{x\to 2}(x+5) = 7$ To be continuous then: $f(x) = \lim_{x\to 2} f(x) = 7$

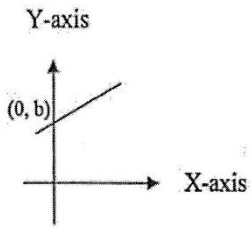
1-3 Equation of straight lines and circles:

A. Equation of Straight lines:



- The slope (m) of a straight line passing through points (x1, y1) & (x2, y2) is:
  $m = \frac{\Delta y}{\Delta x} = \frac{y2-y2}{x2-x1}$ or $\frac{y1-y2}{x1-x2}$

- The equation of a straight line passing through (x1, y1) and has a slop (m) is:
  $y - y1 = m(x - x1)$ ………………. (the point – slop equation of the line)



- The general formula for the equation of a straight line with a slope (m)
  $y - y1 = m(x - x1)$
  $y - b = m(x - 0)$
  $y = mx + b$ …………….. (the point – intercept equation of the line)

__**Note 1:**__ Two lines are parallel if and only if they have the same slopes.

$$L1\ parallel\ L2\ if\ m1 = m2$$

__**Note 2:**__ Two lines are perpendicular if and only if they have the product of their slopes is (-1).

$L1\ perpendicular\ L2\ if\ m1 \times m2 = -1 \rightarrow m1 = \frac{-1}{m2}$

__**Example 1:**__ **Find the equation of the line passing through (-2,-1) & (3,4) ?**

$m = \frac{y2-y1}{x2-x1} = \frac{4+1}{3+2} = 1$

Using $(-2,-1)$ we find:

$y = y1 = m(x - x1)$

$y + 1 = 1(x + 2)$

$y = x + 1$

Or using (3,4) we find: $y - 4 = 1(x - 3)$ $\therefore y = x + 1$

<u>Example 2:</u> Find the slope and y-intercept for $8x + 5y = 20$

$8x + 5y = 20$

$5y = -8x + 20$

$y = \frac{-8}{5}x + 4$

$m = \frac{-8}{5}$ & $b = 4$

<u>Example 3:</u> Find the equation of the line passing through the origin and the point of intersection of → $x + y = 2$ & $L2 \rightarrow 2x - y = -5$?

$x + y = 2 \rightarrow x = 2 - y$

$2(2 - y) - y = -5$

$y = 3 \rightarrow x = -1$

The point of intersection (-1,3)

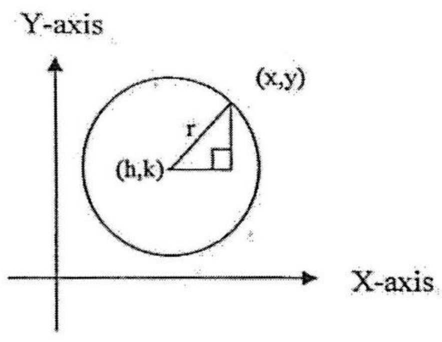
$m = \frac{y2-y1}{x2-x1} = \frac{3+0}{-1+0} = -3$

Using (0,0) or (-1,3) we find:

$y - y1 = m(x - x1)$

$y - 0 = -3(x - 0) \rightarrow y + 3x = 0$

B. Equation of circle:



The equation of a circle centered at (h, k) and has a radius (r) is:

$(x - h)^2 + (y - k)^2 = r^2$

<u>Example 1:</u> Find the radius and coordinate of center for:

$x^2 + y^2 + 4x - 2x - 2y + 1 = 0$

$x^2 + 4x + 4 = 4 + y^2 - 2y + 1 = 0$

$(x + 2)(x + 2) - 4 + (y - 1)(y - 1) = 0$

$$(x+2)^2+(y-1)^2=4$$

The coordinate of center is (-2, 1) and the radius is (2) unit length **Or**

$x^2+y^2+4x-2y+1=0 \quad \rightarrow eq: x^2+y^2+ax+by+c=0$

$h=\frac{-(4)}{2}=-2 \quad \rightarrow h=\frac{-(a)}{2}$

$k=\frac{-(-2)}{2}=1 \quad \rightarrow k=\frac{-(b)}{2}$

$r=\sqrt{(-2)^2+(1)^2-1}=2 \quad \rightarrow r=\sqrt{h^2+k^2-c}$

<u>Example 2:</u> Find the equation of the circle centered at (1, -2) and passing through the point (7, 4) ?

$$d=r=\sqrt{(\Delta x)^2+(\Delta y)^2}$$

$$=\sqrt{(7-1)^2+(4+2)^2}$$

$$=\sqrt{72} \text{ unit length}$$

The equation is:

$$(x-1)^2+(y+2)^2=72$$

<u>Example 3:</u> For this equation: $3y^2-12y+3x^2+6x=18$ find:

- **The center and the radius of the circle.**
- **The equation of the circle.**
- **The area of the circle.**

$3y^2-12+3x^2+6x=18$

$[3x^2+3y^2+6x-12y-18=0]\div 3$

$x^2+y^2+2x-4y-6=0$

$h=\frac{-(a)}{2}=\frac{-(2)}{2}=-1 \;\&\; k=\frac{-(b)}{2}=\frac{-(-4)}{2}=2$

The point of the circle is:

$(x+1)^2+(y-2)^2=11$

The area of circle is: $A=\pi r^2=3.14*11=34.54$ unit area

<u>Ex:</u> Find the slope (m) and y-intercept (b)

1) $8x+5y=20$

$5y=-8x+20$

$y=-\frac{8}{5}x+\frac{20}{5}$

$y=mx+b$

$\therefore m=-\frac{8}{5}$

2) $x-2y=4$

$2y=x-4$

$y=\frac{1}{2}x-\frac{4}{2}$

$m=\frac{1}{2}$

$b=-\frac{4}{2}=-2$

$b = \frac{20}{5} = 4$

<u>Ex:</u> Find the equation of the line passing through the origin and the point of intersection of $L_1: x + y = 2$ and $L_2: 2x - y = -5$

$x + y = 2$

$2x - y = -5$

$3x \quad = -3 \rightarrow x = -1 \rightarrow y = 3 \rightarrow (-1, 3)$

$m = \frac{\Delta y}{\Delta x} = \frac{3-0}{-1-0} = -3$

$y - y_\circ = m(x - x_\circ) \rightarrow y - 0 = -3(x - 0)$

Note:

- Two lines are parallel if and only if they have the same slope $L_1 // L_2$ if $m_1 = m_2$
- Two lines are perpendicular (orthogonal, vertical) if and only if the product of their slopes is (-1)

  $L_1 ⟂ L_2 \; if \; m_1 * m_2 = -1 \; or \; m_1 = -\frac{1}{m_2}$

  A) Circle: the equation of the circle with a center (h, k) and has a radius (r) is $(x - h)^2 + (y - k)^2 = r^2$

  $(x - 2)^2 + (y - k)^2 = 9$ اوجد المركز و نصف القطر

  $3 =$ نق , $(2, -3)$ المركز

  $(x + 2)^2 + (y + 3)^3 = 16$ $\quad 4 =$ نق , $(-2, -3)$ المركز

<u>Ex:</u> Find the radius and coordinate of the center

$x^2 + y^2 + 4x - 2y + 1 = 0$

$x^2 + 4x + y^2 - 2y = -1$

$x^2 + 4x + 4 - 4 + y^2 - 2y + 1 - 1 = -1$ نضيف و نطرح مربع $\frac{1}{2}$ معامل x

$(x + 2)^2 - 4 + (y - 1)^2 - 1 = -1$ نق المركز

$(x + 2)^2 + (y - 1)^2 = 4 \quad \rightarrow \quad (-2, 1), 2$

<u>Ex:</u> For what value of k does the circle $(x - k)^2 + (y - 2k)^2 = 10$ pass through the point $(1, 1)$

$(1 - k)^2 + (1 - 2k)^2 = 10$

$1 - 2k + k^2 + 1 - 4k + 4k^2 - 10 = 0$

$5k^2 - 6k - 8 = 0$

$(5k + 4)(k - 2) = 0$ either $k = -\frac{4}{5}$ or $k = 2$

<u>Ex:</u> Find the equation of the circle centered at (1,-2) and passing through (7,4)

r = المسافة بين نقطتين $\sqrt{(7-1)^2+(4-(-2))^2}=\sqrt{72}$

The equation is $(x-1)^2+(y+2)^2=72$

H.W:

- **Find the equation of the circle that passes through the points A(2, 3), B(-4, 3) and C(3, 2).**
- **Find the equation of the circle which passes through (10, 2), (9, -3) and the center of the circle lies on the y-axis?**

هذا يعني ان احداثيات المركز (*o,k*)

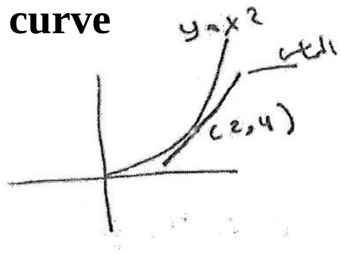
(2) Differentiation

We call that derivate of the function f(x) as $f`(x), \frac{df(x)}{dx}$ $or$ $\frac{dy}{dx}$ $or$ $y`$

- The derivative of a function at a point $x=a$ is the slope of the tangent line to the curve.

عند نقطة التماس ميل المماس = مشتقة الدالة

<u>Ex:</u> Find the equation of the tangent line to the curve $\boldsymbol{y=x^2\ at\ (2,4)}$



$\frac{dy}{dx}=2x$

$m=\frac{dy}{dx}=2*2=4\ at\ (2,4)$

تحقق معادلة المماس (2,4)

$y-y_\circ=m(x-x_\circ)$ $\therefore y-4=4(x-2)$ (2,4) معادلة المماس عند

2-1 Properties of the derivative

1) $\frac{d}{dx}(c)=0 where\ c\ is\ a\ constant$

2) $\frac{d}{dx}[f(x)\pm g(x)]=f`(x)\pm g`(x)$

3) $\frac{d}{dx}\ c\ f(x)=c\ f`(x)$

4) $\frac{d}{dx}[f(x).g(x)]=f(x).g`(x)+g(x).f`(x)$

5) $\frac{d}{dx}\frac{f(x)}{g(x)}=\frac{g(x).f`(x)-f(x).g`(x)}{[g(x)]^2}$

6) $\frac{d}{dx}(x^u)=ux^{u-1}$

7) $\frac{d}{dx}(\sin x) = \cos x$

8) $\frac{d}{dx}(\cos x) = -\sin x$

9) $\frac{d}{dx}(\tan x) = sec^2 x$

10) $\frac{d}{dx}(\cot x) = -csc^2 x$

11) $\frac{d}{dx}(\sec x) = \sec x \tan x$

12) $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Example: Find $\frac{dy}{dx}$ for the following

❖ $y = x^2(x^3 + 2) \Rightarrow y` = x^2(3x^2) + (x^3 + 2) * 2x$

❖ $y = \frac{3}{x} \Rightarrow y` = \frac{x*0-3*1}{x^2} = -\frac{3}{x^2}$

OR $y = 3x^{-1} \Rightarrow y` = -3x^{-2} = -\frac{3}{x^2}$

❖ $y = tan^2 x \Rightarrow y` = 2\tan x \,.\, sec^2 x$

❖ $y = sin^2 x^3 = (\sin x^3)^2$

$y` = 2\sin x^3 \cos x^3 * 3x^2$

❖ $y = \sqrt{\sec(3x^3)} \Rightarrow y = (\sec(3x^3))^{1/2}$

$y` = \frac{1}{2}[\sec(3x^3)]^{-1/2} * \sec 3x^3 \tan 3x^3 * 9x^2$

❖ $y = \sqrt{\sec(x \cos x)} \Rightarrow y = [\sec(x \cos x)]^{1/2}$

$y` = \frac{1}{2}[\sec(x \cos x)]^{-1/2} * \sec(x \cos x) \tan(x \cos x) *$

$[x(-\sin x) + \cos x\,(1)]$

❖ $y = tan^2\left(\cos\frac{1}{x}\right) = \left[\tan\left(\cos\frac{1}{x}\right)\right]^2$

$y` = 2\tan\left(\cos\frac{1}{x}\right).sec^2\left(\cos\frac{1}{x}\right) * -\sin\frac{1}{x} * -x^{-2}$

2-2 Higher order derivative

Find $\frac{d^4y}{dx^4}$ for $y = x^6 - 3x^4 + \cos x$

$y` = 6x^5 - 12x^3 - \sin x$

$y`` = 30x^4 - 36x^2 - \cos x$ $\qquad \frac{d^2x}{dx^2} = y`` = \frac{d}{dx} y` = \frac{d}{dx}\frac{dy}{dx}$

$y``` = 120x^3 - 72x + \sin x$ $\qquad \frac{d^3y}{dx^3} = y``` = \frac{d}{dx}\frac{d^2y}{dx^2}$

$y```` = 360x^2 - 72 + \cos x$

Chain rule:

If $y = f(t)$ and $x = g(t)$ then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

If $y = f(t)$ and $t = g(x)$ then $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

Ex: if $y = \sin t$, $x = \cos t$ find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{-\sin t} = -\cot t$$

Ex: Find $\frac{dy}{dx}$ if $y = t^3$ and $t = x^2 + 2$

$\frac{dy}{dt} = 3t^2 , \frac{dt}{dx} = 2x$

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = 3t^2 * 2x = 3(x^2+2)^2 * 2x$ **Or**

$y = (x^2 + 2)^3$ بالتعويض المباشر

$\frac{dy}{dx} = 3(x^2+2)^2 * 2x$

Ex: Find $\frac{d^2y}{dx^2}$ if $y = (t^2+1)^4 , x = t^2 + 5$

$\frac{dy}{dx} = 4(t^2+1)^3 * 2t , \frac{dx}{dt} = 2t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4(t^2+1)^3 * 2t}{2t} = 4(t^2+1)^3$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{12\left(t^2+1\right)^2 * 2t}{2t} = 12(t^2+1)^2$$

$$\frac{d^2y}{dx^2} = \frac{dy`}{dx} = \frac{\frac{dy`}{dt}}{\frac{dx}{dt}}$$

2-3 Implicit differentiation الاشتقاق الضمني

We differentiate both sides with respect to x

<u>Ex:</u> Find $\frac{dy}{dx}$ if $x^2 + xy + y - x = 0$

$2x + xy` + y * 1 + y` - 1 = 0$

$2x + y - 1 = -xy` - y`$

$2x + y - 1 = y`(-x - 1)$

$y` = \frac{2x+y-1}{-x-1}$

<u>Ex:</u> $\sin y + x \sin x = 1$

$\cos y * y` + x \cos x + \sin x = 0$

$y` \cos y = -x \cos x - \sin x$

$y` = \frac{-x \cos x - \sin x}{\cos y}$

<u>Ex:</u> If $x^2 + y^2 = 1$ show that $\frac{d^2y}{dx^2} = -\frac{1}{y^3}$

$2x + 2y * \frac{dy}{dx} = 0 \Rightarrow 2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$

$$\frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{y+\frac{x^2}{y}}{y^2} = -\frac{\frac{y^2+x^2}{y}}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{y^2+x^2}{y^3} = -\frac{1}{y^3}$$

Ex: Find the equation of the tangent to the curve

$$y = \sin\sqrt{x} \;\; at \; (\pi^2, 0)$$

$$\frac{dy}{dx} = \cos\sqrt{x} * \frac{1}{2\sqrt{x}} = \cos\pi * \frac{1}{2\pi} = -\frac{1}{2\pi} \;\; at \; (\pi^2, 0)$$

$$y - y_\circ = m(x - x_\circ) \Rightarrow y - 0 = -\frac{1}{2\pi}(x - \pi^2)$$

2-4 Application of Derivatives:-

Increasing function when $f`(x) > 0$ المشتقة موجبة

Decreasing function when $f`(x) < 0$ المشتقة سالبة

Horizontal tangent $f`(x) = 0$ عندما

و هذا يعني ان مماس الدالة افقي و ان الدالة تعتبر من متزايدة الى متناقصة او بالعكس.

Ex: Graph the function $y = f(x) = x^3 - 3x^2 + 4$

1) مجد نقاط التقاطع مع المحور

When $x = 0 \Rightarrow y = 4 \Rightarrow (0,4)$

When $y = 0 \Rightarrow 0 = (x+1)(x-2)^2 \Rightarrow x = -1 \Rightarrow (-1,0)$

$x = 2 \Rightarrow (2,0)$

2) نجد الفترات التي تكون فيها المشتقة الاولى موجبة، سالبة، صفر

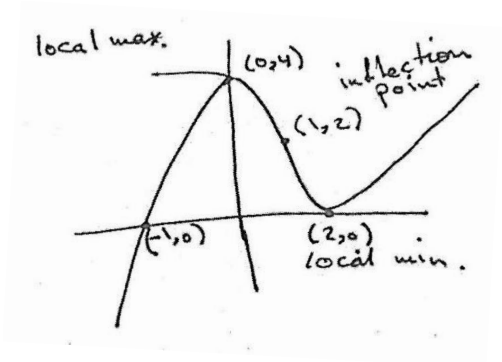
$y` = 3x^2 - 6x = 3x(x-2)$

+ + ● − − ● + +
0 2

3) نجد المشتقة الثانية

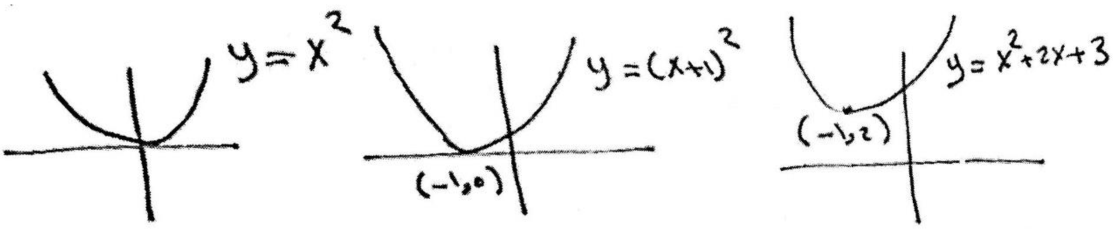
$y`` = 6x - 6 \Rightarrow x = 1$

x	y
-1	0
0	4
1	2
2	0
3	4

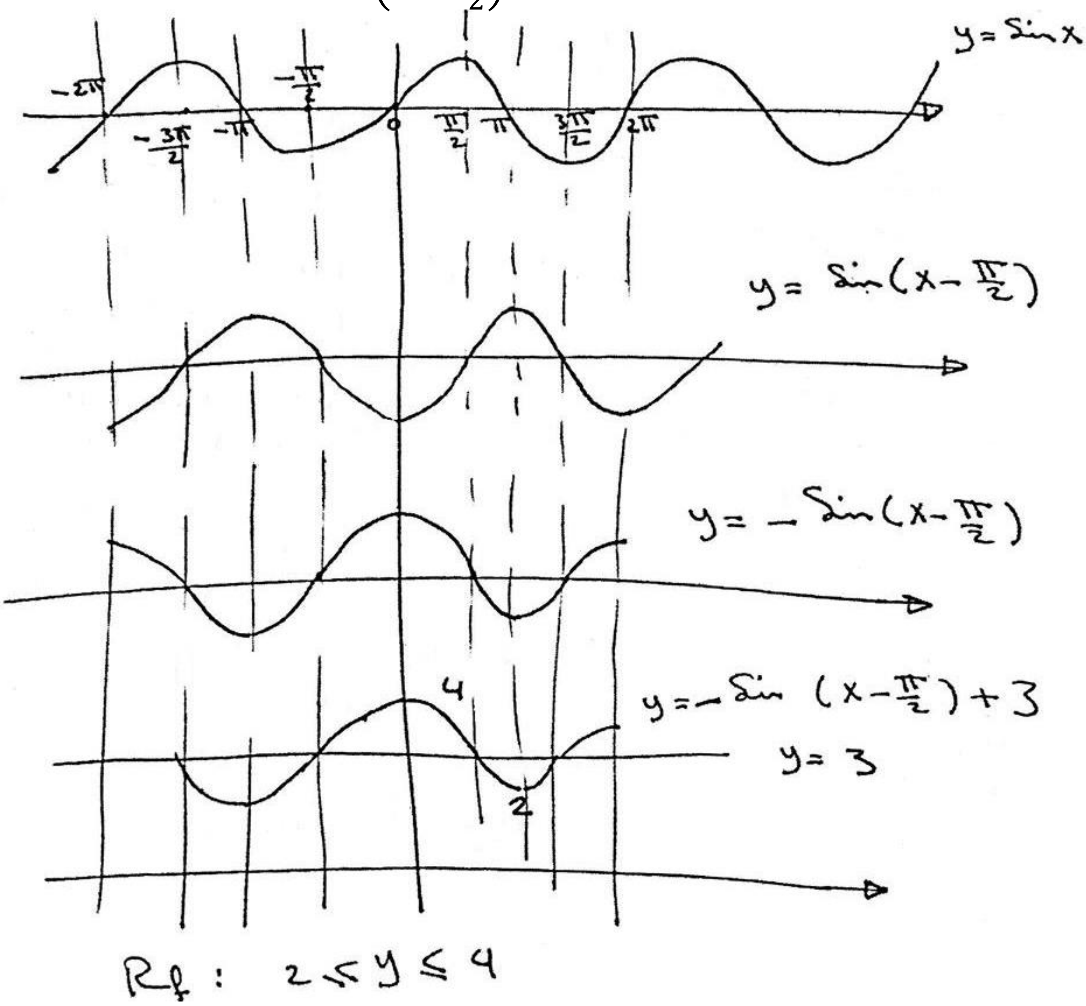


Ex: Graph $y = x^2 + 2x + 3$

$y = x^2 + 2x + 1 - 1 + 3 \Rightarrow y = (x+1)^2 + 2$

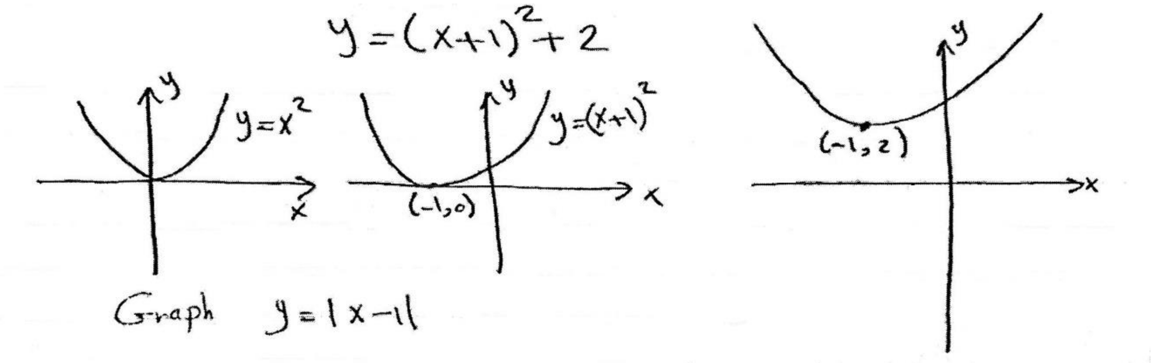


Ex: Graph $y = -\sin\left(x - \frac{\pi}{2}\right) + 3$

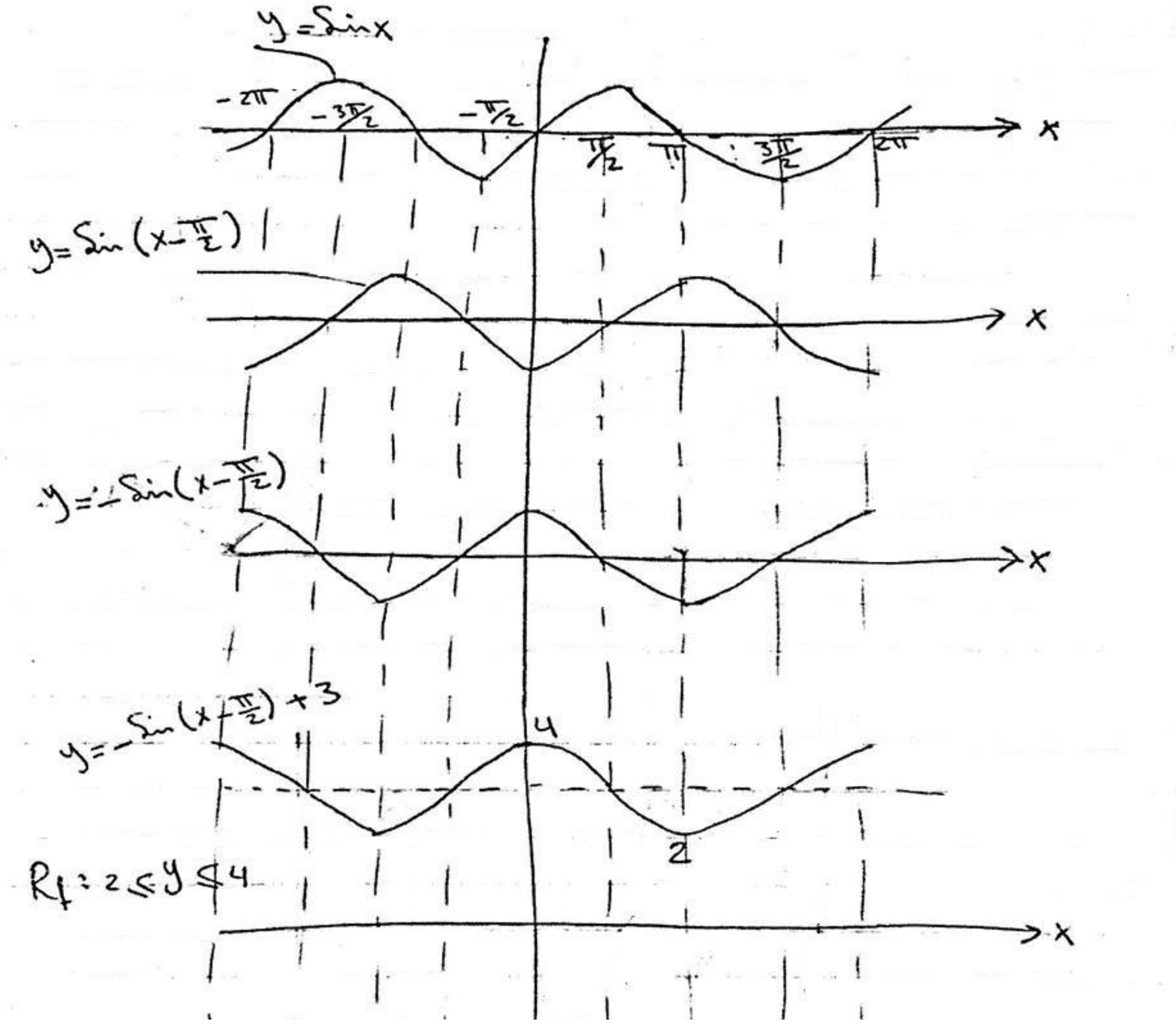


Ex: Graph $y = x^2 + 2x + 3 = x^2 + 2x + 1 - 1 + 3$

$y = (x + 1)^2 + 2$



Ex: Graph $y = -\sin\left(x - \frac{\pi}{2}\right) + 3$



Q1: A body moves in a straight line according to the law of motion

$s = t^3 - 4t^2 - 3t$

Find it's acceleration at each instant when the velocity is zero.

$v = \frac{ds}{dt} = 3t^2 - 8t - 3 \, , a = \frac{dv}{dt} = 6t - 8$

$0 = 3t^2 - 8t - 3 \Rightarrow 0 = (3t + 1)(t - 3)$

$\therefore t = -\frac{1}{3} \; or \; t = 3$

$\therefore a = -10 \; or \; a = 10$

Q2: Find the velocity $v = \frac{ds}{dt}$ and acceleration $a = \frac{dv}{dt}$

1) $s = 2t^3 - 5t^2 + 4t - 3$
2) $s = gt^2/2 + v_\circ t + s_\circ \, , (g, v_\circ, s_\circ \; constants)$
3) $s = (2t + 3)^2$

Q3: Find $y\grave{}$ and $y\grave{}\grave{}$

1) $y = 2x^4 - 4x^2 - 8$
2) $2y = 6x^4 - 18x^2 - 12x$
3) $y = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x + 3$
4) $y = (3x - 1)(2x + 5)$
5) $y = \sin 3x \, . tan^2(x + 2)^2 . \sqrt{x - 1}$
6) $y = \frac{1}{\sec(x^2 y + h)}$ where h is constant

(3) Integration

1) $\int dx = x + c$

2) $\int \big(f(x) \pm g(x)\big)\, dx = \int f(x)dx \pm \int g(x)dx$

3) $\int k.f(x)dx = k \int f(x)dx$ where k is constant

4) $\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad n \neq -1$

5) $\int \frac{dx}{x} = \ln \mid x \mid + c$

<u>Ex:</u> Evaluate the following integrals

1) $\int (x^2 + x)dx = \int x^2 dx + \int xdx = \frac{x^3}{3} + \frac{x^2}{2} + c$

2) $\int 4xdx = 4\int xdx = 4\frac{x^2}{2} + c = 2x^2 + c$

3) $\int (x^2+1)^5.2xdx = \frac{(x^2+1)^6}{6} + c$

4) $\int (x^3 + 3x + 5)(x^2 + 1)dx$

$= \int (x^3 + 3x + 5)(x^2 + 1)\frac{3}{3}dx$

$= \frac{1}{3}\frac{(x^3+3x+3)^2}{2} + c$

5) $\int \frac{x^2+x}{7}dx = \frac{1}{7}\int x^2 dx + \frac{1}{7}\int xdx$

$= \frac{x^3}{21} + \frac{x^2}{14} + c$

6) $\int \frac{dx}{x^5} = \int x^{-5}dx = \frac{x^{-4}}{-4} + c = -\frac{1}{4x^4} + c$

3-1 Integration of trigonometric function

Since $\frac{d}{dx}(\sin x) = \cos x \;\; then \;\; \int \cos x\, dx = \sin x + c$

Similarly for other trigonometric function

1) $\int \cos u \;\, du = \sin u + c$

2) $\int \sin u \;\, du = -\cos u + c$

3) $\int sec^2 u \;\, du = \tan u + c$

4) $\int csc^2 u \;\, du = -\cot u + c$

5) $\int \sec u\, tanu\, du = \sec u + c$

6) $\int \csc u \cot u\, du = -\csc u + c$

7) $\int \tan u \;\, du = \int \frac{\sin u}{\cos u} dx = -\ln|\cos x| \; + c$

8) $\int \cot x \;\, dx = \int \frac{\cos x}{\sin x} dx = \ln|\sin x| + c$

9) $\int \sec x \;\, dx = \ln|\sec x + \tan x| + c$

10) $\int \csc x\, dx = \ln|\csc x - \cot x| + c$

<u>Ex:</u>

1) $\int \cos 3x\, dx = \frac{1}{3}\int \cos 3x * 3x dx = \frac{1}{3}\sin 3x + c$

2) $\int \sin 7x\, dx = -\frac{1}{7}\cos 7x + c$

3) $\int sec^2(x+3) dx = \tan(x+3) + c$

4) $\int \cot 2x \,.\, \csc 2x \, dx = -\frac{1}{2}\int -\csc 2x \cot 2x * 2dx$

$$= -\frac{1}{2}\csc 2x + c$$

5) $\int x \sin(2x^2)\, dx = \frac{1}{4}\int \sin(2x^2) * 4x\, dx$

$$= -\frac{1}{4}\cos 2x^2 + c$$

6) $\int 2 \sin x cos\; x\, dx = 2\int(\sin x)\cos x\, dx$

$= 2\frac{sin^2 x}{2} + c = sin^2 x + c$

Definite integrals:-

$\int_a^b f(x)\, dx$

a is called the lower bound

b is called the cupper bound

properties of definite integrals:-

1) $\int_a^a f(x)\;\, dx = 0$

<u>Ex:</u> $\int_3^3 \frac{x^3+3x^2-2}{cos^3 x} dx = 0$

2) $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

<u>Ex:</u> $\int_1^3 2x\, dx = -2\int_3^1 x\, dx$

3) $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$

<u>Ex:</u> $\int_1^3 2x\, dx = \int_1^2 2x\, dx + \int_2^3 2x\, dx$

4) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

5) $\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

6) If (f) is even function then

$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

<u>Ex:</u> $\int_{-4}^{4}(x^2-5)\,dx = 2\int_0^4 (x^2-5)\,dx$

7) If (f) is odd function then

$\int_{-a}^{a} f(x)\,dx = 0$

$\int_{-4}^{4} 2x\,dx = 0$

Find the curve whose slope at the point (x,y) is $3x^2$ if it passes through point (1,-1)

$\frac{dy}{dx} = 3x^2 \Rightarrow \int dy = \int 3x^2\,dx$

$y = x^3 + c$

$-1 = 1 + c \Rightarrow c = -2$

$y = x^3 - 2$

Find the curve whose slope at the point (x,y) is $x\sqrt{x^2-1}$ and passes through point (2,3)

$y = \int x\sqrt{x^2-1}\,dx \Rightarrow y = \frac{1}{2}\int (x^2-1)^{1/2} * 2x$

$y = \frac{1}{3}(x^2 - 1)^{3/2} + c$

$3 = \frac{1}{3}(4 - 1)^{3/2} + c \Rightarrow 3 = \frac{1}{3}(3)^{3/2} + c$

$c = 1.268$

Find the velocity of acceleration of a particle whose position function is

$s(t) = \frac{1}{t^2+2t+1}$ when t=2

Examples:

1) $\int_{-1}^{1}(2x^2 - x^3)\,dx = \frac{2x^3}{3} - \frac{x^4}{4}\,]_{-1}^{1}$

$= \left(\frac{2}{3} - \frac{1}{4}\right) - \left(\frac{-2}{3} - \frac{1}{4}\right) = \frac{2}{3} - \frac{1}{4} + \frac{2}{3} + \frac{1}{4} = \frac{4}{3}$

2) $\int_{-3}^{-1} \frac{1}{x^2} - \frac{1}{x^3}\,dx = \left[-\frac{1}{x} + \frac{1}{2x^2}\right]_{-3}^{-1}$

$\left(1 + \frac{1}{2}\right) - \left(\frac{1}{3} + \frac{1}{18}\right) = \frac{10}{9}$

3) $\int_{1}^{4} \frac{dx}{\sqrt{x}} = \int_{1}^{4} x^{-\frac{1}{2}}\,dx = 2\sqrt{x}]_{1}^{4} = 2(\sqrt{4} - \sqrt{1}) = 2$

4) $\int_{0}^{2}(2 - x)^2 dx = \int_{0}^{2} 4 - 4x + x^2 dx$

$= 4x - 2x^2 + \frac{x^3}{3}\,|_{0}^{2} = \left(8 - 8 + \frac{8}{3}\right) = \frac{8}{3}$

Evaluate the following integrals:-

1) $\int \frac{\cos 2x}{sin^3 2x} dx = \int \cos 2x * (sin2x)^{-3} dx * \frac{2}{2}$

$= \frac{1}{2}\frac{(\sin 2x)^{-2}}{-2} + c$

Or $\int \frac{\cos 2x}{sin^3 2x} dx = \int \frac{\cos 2x}{\sin 2x} * \frac{1}{sin^2 2x} dx = \int \cot 2x * csc^2 2x\, dx$

$$= -\frac{1}{2}\frac{(\cot 2x)^2}{2} + c$$

Or $\int \frac{\cos 2x}{sin^3 2x} = \int \cot 2x * \csc 2x * \csc 2x\, dx * \frac{-2}{-2}$

$= -\frac{1}{2}\frac{(\csc 2x)^2}{2} + c$ Or طريقة الفرضية

Let $y = \sin 2x \Rightarrow dy = \cos 2x * 2\, dx$

$$\frac{dy}{2} = \cos 2x * dx$$

$\int \frac{\cos 2x}{sin^3 2x} dx = \int \frac{dy/2}{y^3} = \frac{1}{2}\int \frac{dy}{y^3} = \frac{1}{2}\int y^{-3}\, dy$

$= \frac{1}{2}\frac{y^{-2}}{-2} + c = \frac{1}{2}\frac{(\sin 2x)^{-2}}{-2} + c$

2) $\int \frac{\cos x\, dx}{\sqrt{1-\sin x}} = \int \frac{\cos x}{(1-\sin x)^{1/2}} = \int (1-\sin x)^{-\frac{1}{2}} \cos x\, dx$

$= -\frac{(1-\sin x)^{1/2}}{1/2} + c$

3) $\int \sqrt{1-\sin x}\, dx * \frac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}} = \int \frac{\sqrt{1-sin^2 x}}{\sqrt{1+\sin x}} dx$

$= \int (1+\sin x)^{-\frac{1}{2}} * \cos x\, dx = \frac{(1+\sin x)^{1/2}}{1/2} + c$

4) $\int cos^3 3x\, dx = \int cos^2 3x . \cos 3x\, dx$

$= \int (1 - sin^2 3x) * \cos 3x\, dx$

$= \int \cos 3x\, dx - \int sin^2 3x \cos 3x\, dx$

$$= \frac{1}{3}\sin 3x - \frac{1}{3}\frac{sin^3 3x}{3} + c$$

Evaluate the following integration:

1) $\int \frac{(Z+1)\,dZ}{\sqrt[3]{Z^2+2Z+2}} = \frac{1}{2}\int 2(Z+1)(Z^2+2Z+2)^{-\frac{1}{3}}\,dZ$

$$= \frac{1}{2} * \frac{(Z^2+2Z+2)^{\frac{2}{3}}}{\frac{2}{3}} + c$$

2) $\int \frac{d\theta}{sin^2\frac{\theta}{3}} = \int csc^2\frac{\theta}{3}\,d\theta$ $\qquad \int csc^2\theta = -\cot\theta$

$$= -3\int -\frac{1}{3}csc^2\frac{\theta}{3}\,d\theta = -3\cot\frac{\theta}{3} + c$$

3) $\int \sqrt{\frac{sin^2x}{1+\cos x}}\,dx = \int \frac{\sqrt{sin^2x}}{\sqrt{1+\cos x}}\,dx$

$$= -\int -\sin x\,(1+\cos x)^{-\frac{1}{2}}\,dx$$

$$= -\frac{(1+\cos x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

4) $\int \frac{\sqrt{\cot x}}{sin^2x}\,dx = -\int (\cot x)^{\frac{1}{2}} * -csc^2x\,dx$

$$= -\frac{(\cot x)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

5) $\int \sin(\tan\theta)sec^2\theta\,d\theta$ $\qquad \int \sin x\,dx = -\cos x + c$

$= -\cos(\tan\theta) + c$ $\qquad \frac{1}{2}\int 2x\sin x^2\,dx = \frac{1}{2} * -\cos x^2$

6) $\int (\tan\theta + \cot\theta)^2 \, d\theta$

$= \int (tan^2\theta + 2\tan\theta\cot\theta + cot^2\theta) \, d\theta$

$= \int tan^2\theta \, d\theta + 2\int d\theta + \int cot^2\theta \, d\theta$

$= \int (sec^2\theta - 1) \, d\theta + 2\int d\theta + \int (csc^2\theta - 1) \, d\theta$

$= \tan\theta - \theta + 2\theta - \cot\theta - \theta + c$

$= \tan\theta - \cot\theta + c$

$\cos 2x = \cos(x + x) = \cos x \cos x - \sin x \sin x$

$\cos 2x = cos^2 x - sin^2 x$

$\cos 2x = 1 - 2sin^2 x \rightarrow sin^2 x = \frac{1 - \cos 2x}{2}$

$\sin 2x = 2\sin x \cos x$ $\quad$ $\sin(x + x) = \sin x \cos x + \cos x \sin x$

7) $\int \sin 2t \, (cos^2 t - sin^2 t) \, dt$

$= \int \sin 2t \cos 2t \, dt = \frac{1}{2}\int 2(\sin 2t)\cos 2t \, dt$

$= \frac{1}{2}\frac{(\sin 2t)^2}{2} + c$

8) $\int sin^2\frac{\theta}{2} \, d\theta = \int \frac{1 - \cos\theta}{2} d\theta$

$= \frac{1}{2}\int (1 - \cos\theta) \, d\theta = \frac{1}{2}\int d\theta - \frac{1}{2}\int \cos\theta \, d\theta$

$= \frac{1}{2}\theta - \frac{1}{2}\sin\theta + c$

9) $\int \frac{\tan\theta}{\cos x\sqrt{1+\sec x}} dx$

$\int \sec x \tan x \,(1+\sec x)^{-\frac{1}{2}} \, dx$

$= \frac{(1+\sec x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

10) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx = \int x^{\frac{1}{2}} \, dx + \int x^{-\frac{1}{2}} \ d$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

Q1: Evaluate the following integrals:-

1) $\int cos^2 x \,(2+\sin x)\, dx = \int 2cos^2 x \ dx + \int cos^2 x \sin x \, dx$

$= \int 2\,\frac{1+\cos 2x}{2} dx + \int cos^2 x \sin x \, dx$

$= x + \frac{1}{2}\sin 2x - \frac{cos^3 2x}{6} + c$

2) $\int sec^3 2x \tan 2x \, dx = \frac{1}{2}\int sec^2 2x * \sec 2x * \tan 2x * 2 \, dx$

$= \frac{1}{2}\frac{sec^3 2x}{3} + c = \frac{sec^3 2x}{6} + c$

3) $\int sec^2 \frac{x}{3}\left(1 + tan^2 \frac{x}{3}\right) dx = \int sec^2 \frac{x}{3} dx + \int sec^2 \frac{x}{3} tan^2 \frac{x}{3} \ dx$

$= 3\tan\frac{x}{3} + 3\,\frac{tan^3 x/3}{3} + c = 3\tan\frac{x}{3} + tan^3 \frac{x}{3} + c$

4) $\int \frac{dx}{cos^2\frac{x}{2}\left(2+3\tan\frac{x}{2}\right)^4} = \frac{2}{3}\int \left(2 + 3\tan\frac{x}{2}\right)^{-4} * sec^2 \frac{x}{2} \ dx * 3 * \frac{1}{2}$

$$= \frac{2}{3} * \frac{\left(2+3\tan\frac{x}{2}\right)^{-3}}{-3} = -\frac{2}{9}\left(2+3\tan\frac{x}{2}\right)^{-3} + c$$

Q2: Find $\frac{dy}{dx}$

1) $sin^2 xy + \tan y = x^2 - y \Rightarrow 2\sin xy\,.\cos xy\,.(xy` + y) + sec^2 y.y` = 2x - y`$

$$2xsin\ xy.\cos xy\,.* y` + 2y\sin xy\cos xy + sec^2 y.y` = 2x - y`$$

$$y`(2xsin\ xy\cos xy) + sec^2 y + 1 = 2x - 2ysin\ xy\cos xy$$

$$y` = \frac{2x - 2y\sin xy\cos xy}{2x\sin xy\cos xy + sec^2 y + 1}$$

2) $sec^3 y + \cot(xy) = x^3 - y^2 \Rightarrow 3sec^2 y.\sec y\tan y\,.y`$

$$-csc^2(xy)(xy` + y) = 3x^2 - 2yy`$$

$$\Rightarrow (3sec^2 y.\sec y\,.\tan y - xcsc^2 xy + 2y)y` = ycsc^2 xy + 3x^2$$

$$y` = \frac{ycsc^2 xy + 3x^2}{3sec^3 y.\tan y - xcsc^2 xy + 2y}$$

Q3: Calculate $\lim_{x\to 2} \frac{(x^2+x-6)\tan(x-2)}{(x-2)^2}$

$$= \lim_{x\to 2} \frac{(x-2)(x+3)\frac{\sin(x-2)}{\cos\,(x-2)}}{(x-2)^2}$$

$$= \lim_{x\to 2} \frac{(x+3)}{\cos(x-2)} \boxed{\frac{\sin(x-2)}{x-2}} \Rightarrow 1 \qquad \therefore = \frac{2+3}{\cos(0)} = \frac{5}{1} = 5$$

MATRICES AND DETERMINANTS

المصفوفات و المحددات

1) MATRICES ت

المصفوفة (matrix) هي مجموعة من الاعداد الحقيقة أو الاعداد المركبة أو العناصر، مرتبة في صفوف (rows) واعمدة (columns) و بذلك تشكل حيز متسطيل مثل:

$\begin{bmatrix} 5 & 7 & 2 \\ 6 & 3 & 8 \end{bmatrix}$: This is called a 2*3 matrix because it has (2) rows and (3) columns

- مرتبة المصفوفة (order) يشير الى عدد صفوفها واعمدتها و يرمز له (m*n) حيث ان (m) هي عدد الصفوف و (n) عدد الاعمدة.
- في المصفوفات يتم حصر الارقام (أو العناصر) بين اقواس مربعة كبيرة [ ]
- المصفوفة هي حيز من الارقام التي قد لا توجد علاقة رياضية بينهم، و بذلك فأنه لا يوجد ناتج عددي للمصفوفة (اي ان المصفوفة لا تساوي عدد ما وانما هي فقط ترتيب معين من الارقام).
- الترميز المزدوج (Double suffix notation)

لكي يتم تمييز موقع عنصر ما في المصفوفة فأنه يتم استخدام الرمز المزدوج مثل (a_{23}) حيث ان (a) هي العنصر (الرقم). والرقم (2) يشير الى الصف الثاني و الرقم (3) يشير الى العمود الثالث و بذلك فأن موقع هذا العنصر يكون في الصف الثاني، العمود الثالث من المصفوفة.

Ex: $\begin{bmatrix} 6 & -5 & 1 & -3 \\ 2 & -4 & 8 & 3 \\ -2 & 9 & 7 & -1 \end{bmatrix}$ the order is $(3 * 4)$

$$a_{24} = 3 \quad a_{33} = 7 \quad a_{22} = -4$$
$$a_{14} = -3 \quad a_{23} = 8 \quad a_{11} = 6 \qquad \textit{and so on}$$

- يمكن ان نعطي رمز للمصفوفة مثل حرف A أو حرف B أو حرف C ..... الخ لكي يتم تمييز مصفوفة عن الاخرى مثلا :-

[A] = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, [B] = $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$

∴ [A] is a 2*3 matrix and [B] is a 3*2 matrix

❖ ***Equal Matrices:*** **المصفوفات المتساوية**

Two matrices are said to be equal if corresponding elements throughout are equal. Therefore, the two matrices must be of the same order.

❖ **جمع و طرح المصفوفات (Addition And Subtraction)**

لكي يتم جمع أو طرح مصفوفتين يجب ان تكونا اولا من نفس المرتبة (order). فأذا كانتا كذلك فأنه يتم جمع أو طرح العناصر المتقابلة (المتناظرة في الموقع) في المصفوفتين.

Ex: If A = $\begin{bmatrix} 4 & 2 & 3 \\ 5 & 7 & 6 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 8 & 9 \\ 3 & 5 & 4 \end{bmatrix}$

∴ A+B = $\begin{bmatrix} 4+1 & 2+8 & 3+9 \\ 5+3 & 7+5 & 6+4 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 12 \\ 8 & 12 & 10 \end{bmatrix}$

And B – A = $\begin{bmatrix} 1-4 & 8-2 & 9-3 \\ 3-5 & 5-7 & 4-6 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 6 \\ -2 & -2 & -2 \end{bmatrix}$

❖ ضرب المصفوفات (**Multiplication of Matrices**) :-

أ. ضرب المصفوفة برقم ثابت : يضرب الرقم في جميع عناصر المصفوفة.

Ex: $4 * \begin{bmatrix} 3 & 2 & 5 \\ 6 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 12 & 8 & 20 \\ 24 & 4 & 28 \end{bmatrix}$

ب. ضرب مصفوفة في مصفوفة : يمكن ضرب مصفوفة في اخرى فقط اذا توفر الشرط التالي :- "عدد اعمدة المصفوفة الاولى = عدد صفوف المصفوفة الثانية" . و اذا لم يتوفر هذا الشرط فلا يمكن ضرب المصفوفتين.

Ex: $[A]_{3*2}$, $[B]_{2*4}$

∴ A ∗ B is possible [since (3*2)*(2*4)] the dimensions of the new matrix will be 3*4

$B * A$ is not possible [since (2*4)*(3*2)]

لا يتوفر الشرط لذلك لا يمكن ضرب المصفوفتين بهذا الترتيب.

❖ اذا توفر الشرط السابق يمكن ضرب المصفوفتين، و يتم اجراء الضرب كالتالي:

- تضرب عناصر الصف الاول (من المصفوفة الاولى) في عناصر العمود الاول (من المصفوفة الثانية) . ويتم وضع الناتج في مصفوفة جديدة في الموقع (a_{11}) (حاصل جمع الضرب).
- نضرب عناصر الصف الثاني (من المصفوفة الاولى) في عناصر العمود الاول (من المصفوفة الثانية). ويتم وضع الناتج في المصفوفة الجديدة في الموقع (a_{21}) (حاصل جمع الضرب).
- نستمر بنفس الاسلوب الى ان تكتمل جميع عناصر المصفوفة الجديدة.

Ex: A = $\begin{bmatrix} 4 & 7 & 6 \\ 2 & 3 & 1 \end{bmatrix}$, B = $\begin{bmatrix} 8 \\ 5 \\ 9 \end{bmatrix}$

بما ان المصفوفة [A] هي (2*3) و المصفوفة [B] هي (3*1)، اذا توفر الشرط

∴ [A]*[B] = (2*3)*(3*1)

الشرط

المصفوفة الناتجة الجديدة

حيث تساوي عدد الاعمدة في المصفوفة الاولى مع عدد الصفوف في المصفوفة الثانية و بذلك ستنتج مصفوفة جديدة ذات مرتبة (2*1).

$$[A*B] = \begin{bmatrix} (4)(8)+(7)(5)+(6)(9) \\ (8)(2)+(3)(5)+(1)(9) \end{bmatrix} = \begin{bmatrix} 121 \\ 40 \end{bmatrix}$$

Ex: [A] = $\begin{bmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 4 \end{bmatrix}$, [B] = $\begin{bmatrix} 8 & 4 & 3 & 1 \\ 2 & 5 & 8 & 6 \end{bmatrix}$

$$\therefore [A * B]$$

$$= \begin{bmatrix} (8)(1)+(5)(2) & (4)(1)+(5)(5) & (3)(1)+(5)(8) & (1)(1)+(5)(6) \\ (8)(2)+(7)(2) & (4)(2)+(5)(7) & (3)(2)+(7)(8) & (1)(2)+(7)(6) \\ (8)(3)+(4)(2) & (4)(3)+(5)(4) & (3)(3)+(4)(8) & (1)(3)+(4)(6) \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 29 & 43 & 31 \\ 30 & 43 & 62 & 44 \\ 32 & 32 & 41 & 27 \end{bmatrix}$$

❖ المصفوفة المحولة (Transpose Matrix)

اذا تم تبديل مواقع الاعمدة في مصفوفة ما بمواقع الصفوف لنفس المصفوف اي ان الصف الاول يصبح العمود الاول و الصف الثاني يصبح العمود الثاني و الصف الثالث يصبح العمود الثالث و هكذا .....فأن المصفوفة الجديدة المتشكلة من هذا التحويل تدعى (Transpose matrix) و يرمز لها بالرمز (A^T) اذا كانت المصفوفة الاولية هي (A).

Ex: If $A = \begin{bmatrix} 4 & 6 \\ 7 & 9 \\ 2 & 5 \end{bmatrix}$ then $A^T = \begin{bmatrix} 4 & 7 & 2 \\ 6 & 9 & 5 \end{bmatrix}$

Ex: (H.W) if $A = \begin{bmatrix} 2 & 7 & 6 \\ 3 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 3 & 7 \\ 1 & 5 \end{bmatrix}$ then find: $(A * B)^T$

$$\left(ans. \begin{bmatrix} 35 & 20 \\ 79 & 32 \end{bmatrix}\right)$$

❖ Special matrices:

a) Square matrix: is a matrix of order m*m

المصفوفة المربعة: هي المصفوفة التي يتساوي فيها عدد الاعمدة مع عدد الصفوف.

b) Diagonal matrix: is a square matrix with all elements zero except these on the leading diagonal.

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix} \text{ is a diagonal matrix}$$

c) Unit matrix: is a diagonal matrix in which the elements on the leading diagonal are all unity i.e. :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ is a unit matrix and denoted by } I$$

$$Note : A.I = I.A = A$$

d) Null matrix: a null matrix is a matrix whose elements are all zero

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2) DETERMINANTS المحددات

With each square matrix A we associate a number called the determinant of [A] (denoted by det. A or $|a_{ij}|$). The determinant of matrix is calculated from the entries

(a_{11}, a_{12}, a_{13},…..) of A.

- من كل مصفوفة مربعة (مثل المصفوفة A) يمكن ان نستخرج رقم يدعى (محددة المصفوفة) و ذلك بأن نعامل عناصر المصفوفة (a_{11} , a_{12} , a_{13}) بطريقة معينة لاستخراج هذا المحدد. وبصورة عامة يمكن ايجاد المحدد بطريقة Minors and cofactors.

- **Minors and cofactors:**

a) Minors: the minor of the element a_{ij} [where (i) is the number of row, and (j) is the number of column] is a matrix [A] in the determinant of the matrix which remains after the row and the column containing the element a_{ij} are deleted.

- لكل عنصر من عناصر مصفوفة ما (مثل المصفوفة A) هنالك محدد تدعى (minor) لعنصر المعين. هذا ال (minor) يمكن ايجاده من خلال حذف الصف و العمود الذين يحتويان العنصر المعين . المحدد المتبقة بعد الحذف تدعى (minor) العنصر.

<u>Ex:</u> If $[A]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $\det A=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

<u>ملاحظة:</u> يتم تمييز المصفوفة عن المحددة بالاقواس التي تحيط بالعناصر. فالمصفوفة اقواسها مربعة كبيرة [ ] ، و المحددة اقواسها خطوط مستقيمة | | .

The minor of a_{23} is : $\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$ The minor of a_{11} is : $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

The minor of a_{33} is : $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ The minor of a_{12} is : $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

and so on.

b) Cofactors: the cofactor of a_{ij}, is the determinant (A_{ij}) that is $(-1)^{i+j}$ times the minor of a_{ij} :-

$$A_{ij} = (-1)^{i+j} * minor\ of\ a_{ij}$$

❖ اعطاء اشارة (سالبة او موجبة) الى (minor) عنصر معين يؤدي الى حصولنا على (cofactor) ذلك العنصر (يرمز له a_{ij}) . الاشارة تعتمد على موقع العنصر في المصفوفة و تتبع القاعدة اعلاه أو الجدول ادناه:

$$\begin{vmatrix} + & - & + & \cdot & \cdot \\ - & + & - & \cdot & \cdot \\ + & - & + & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

<u>Ex:</u> if $[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

➢ The minor of $a_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

The cofactor of a_{31} , $A_{31} = (-1)^{3+1} * \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

➢ The minor of $a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

The cofactor of a_{23} , $A_{23} = (-1)^{2+3} * \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$ ……. And so on.

❖ We can find the value of the determinant by summing the (cofactors*entry) of the entries of any one row <u>or</u> any one column, thus:-

يمكن ايجاد قيمة المحدد لاي مصفوفة مربعة باستخدام اي صف او اي عمود (ويفضل استخدام الصف او العمود الذي يحتوي اكبر قدر ممكن من الاصفار لتسهيل العمليات الحسابية)

<u>Ex:</u> $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

❖ نأخذ الصف الاول مثلا:

$$= a_{11}.A_{11} + a_{12}.A_{12} + a_{13}.A_{13}$$

$$= a_{11}.(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}.(-1)^{1+2}.\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}.(-1)^{1+3}.\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

❖ و اذا تم اختيار العمود الثالث مثلا:-

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{13}.(-1)^{1+3}.\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} + a_{23}.(-1)^{2+3}.\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$
$$+ a_{33}.(-1)^{3+3}.\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

و هكذا الحال اذا تم اختيار اي عمود أو أي صف.

❖ EXPANDING AND EVALUATING A DETERMINANT:

- For a matrix with only one element, det [a] = a
- For a matrix with 2*2 , det $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}.a_{22} - a_{21}.a_{12}$
- For a matrix with 3*3

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$= (a_{11}.a_{22}.a_{33}) + (a_{12}.a_{23}.a_{31}) + (a_{13}.a_{21}.a_{32}) - (a_{31}.a_{22}.a_{13})$$
$$- (a_{32}.a_{23}.a_{11}) - (a_{33}.a_{21}.a_{12})$$

<u>ملاحظة مهمة :</u> هذه الطريقة في فتح المحدد خاصة بمحددة 3*3 و لا يجوز استخدامها مطلقا اذا كانت المحددة اكبر من 3*3 .

❖ For a matrix more than 3*3 we use the definition of (minor) and (cofactors).

اذا كانت المصفوفة المربعة اكبر من 3*3 فيجب ان نستخدم نعاريف (minor) و (cofactor) لكي يتم ايجاد المحددة، لان استخدام ال(cofactor) يقلل درجة (مرتبة) المحددة.

<u>Ex:</u> Evaluate $A = \begin{vmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$ حل السؤال سيتم باكثر من طريقة

أ) الاسلوب الاول : بأستخدام (minor) و (cofactor) و اختيار الصف الاول:

$$\det A = a_{11}.A_{11} + a_{12}.A_{12} + a_{13}.A_{13}$$

$$= 2 * (-1)^2 * \begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} + (1) * (-1)^3 * \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} + (3)(-1)^4 * \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= 2(-1+6) - 1 * (3+4) + 3(9+2) = 36$$

ب) الاسلوب الثاني : بأستخدام (minor) و (cofactor) و اختيار العمود الثالث:

$\det A = a_{13}.A_{13} + a_{23}.A_{23} + a_{33}.A_{33}$

$= 3\,(-1)^4 \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} + (-2)(-1)^5 \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + (1)(-1)^6 \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = 36$

ت) الاسلوب الثالث : بأٍتخدام الطريقة الخاصة للمحدد 3* 3

$$\left|\begin{array}{ccc|cc} 2 & 1 & 3 & 2 & 1 \\ 3 & -1 & -2 & 3 & -1 \\ 2 & 3 & 1 & 2 & 3 \end{array}\right| =$$

$$= (2)(-1)(1) + (1)(-2)(2) + (3)(3)(3) - (2)(-1)(3) - (3)(-2)(2) - (1)(3)(1)$$

$$= -2 - 4 + 27 + 6 + 12 - 3 = 36$$

<u>Ex:</u> Find the value of (x) if : $\begin{vmatrix} 1 & 1 & -1 \\ x & 1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = 0$

❖ Expanding first row:

$$1.(-1)^2.\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + (1)(-1)^3.\begin{vmatrix} x & 2 \\ 1 & -1 \end{vmatrix} + (-1)(-1)^4.\begin{vmatrix} x & 1 \\ 1 & 2 \end{vmatrix} = 0$$

$$= (-1-4) - (-x-2) - (2x-1) = 0$$

$$-5 + x + 2 - 2x + 1 = 0 \longrightarrow x = -2$$

Or :

$$\left|\begin{array}{ccc|cc} 1 & 1 & -1 & 1 & 1 \\ x & 1 & 2 & x & 1 \\ 1 & 2 & -1 & 1 & 2 \end{array}\right| = 0$$

=(1*1*-1 + 1*2*1 + -1*x*2) – (-1*1*1+1*2*2 +1*x*-1) =0

-1+2 -2x +1- 4+ x = 0 ⟶ x = -2

❖ CRAMER'S RULE

Is a rule for solving a system of linear equations, like three equations with three unknowns

If $a_{11}.x + a_{12}.y + a_{13}.Z = b_1$ ....(1)

$$a_{21}.x + a_{22}.y + a_{23}.Z = b_2 \text{ ....(2)}$$

$$a_{31}.x + a_{32}.y + a_{33}.Z = b_3 \text{ ....(3)}$$

Then cramer's rule gives:-

$$x = \frac{Dx}{D} \quad , \quad y = \frac{Dy}{D} \quad , \quad Z = \frac{DZ}{D}$$

Where ; $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, $D_x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$

$$D_y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} , \quad D_Z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Ex: Find the values of x , y and Z by cramer's rule for:

$$y - Z = 3$$

$$2x - 2Z = 2$$

$$2y + Z = 3$$

Sol: Re- arranged the equation as

$$0.x + 1.y - 1.Z = 3$$

$$2.x + 0.y - 2.Z = 2$$

$$0.x + 2.y + 1.Z = 3$$

$$\therefore D = \begin{vmatrix} 0 & 1 & -1 \\ 2 & 0 & -2 \\ 0 & 2 & 1 \end{vmatrix} = -6 \ , \quad D_x = \begin{vmatrix} 3 & 1 & -1 \\ 2 & 0 & -2 \\ 3 & 2 & 1 \end{vmatrix} = zero$$

$$D_y = \begin{vmatrix} 0 & 3 & -1 \\ 2 & 2 & -2 \\ 0 & 3 & 1 \end{vmatrix} = -12 \ , \quad D_Z = \begin{vmatrix} 0 & 1 & 3 \\ 2 & 0 & 2 \\ 0 & 2 & 3 \end{vmatrix} = 6$$

$$\therefore x = \frac{D_x}{D} = \frac{0}{-6} = 0 \quad , \quad y = \frac{D_y}{D} = \frac{-12}{-6} = 2 \quad , \quad Z = \frac{D_z}{-6} = \frac{6}{-6} - 1$$

EX: solve the following system of equation by cramer's rule;

$$(t-1).x + t.y = 1$$

$$t.x + (t-1).y = -1$$

$$D = \begin{vmatrix} t-1 & t \\ t & t-1 \end{vmatrix} = (t-1)^2 - t^2 = t^2 - 2t + 1 - t^2 = -2t + 1$$

$$D_x = \begin{vmatrix} 1 & t \\ -1 & t-1 \end{vmatrix} = t - 1 + t = 2t - 1 \Rightarrow x = \frac{D_x}{D} = \frac{2t-1}{-2t+1}$$

$$= -1$$

$$D_y = \begin{vmatrix} t-1 & 1 \\ t & -1 \end{vmatrix} = -t + 1 - t = -2t + 1 \Rightarrow y = \frac{D_y}{D} \Rightarrow$$

$$y = \frac{-2t+1}{-2t+1} = 1$$

(H.W) Find the values of (x) if : $\begin{vmatrix} x & 2 & 3 \\ 2 & x+3 & 6 \\ 3 & 4 & x+6 \end{vmatrix} = 0$

Transcendental functions

1) Logarithmic functions
   a) Natural logarithmic functions:-

$\log_e m = x \; then \quad m = e^x$

where e is a real number ($e = 2.718$)

$$\log_e m = x \quad shall\ be\ written\ as \quad \ln m = x$$

   b) Common logarithmic functions:-

$$\log_{10} m = x \; then \; m = 10^x$$

$$\log_{10} m = x \quad shall\ always\ written\ as \quad \log m = x$$

$$\therefore \log_{10} m = \log m = x$$

   c) General logarithmic functions:-

$$\log_a m = x \quad then \quad m = a^x$$

Properties of logarithmic functions:-

1. $\log(x.y) = \log x + \log y$
2. $\ln(x.y) = \ln x + \ln y$
3. $\log \frac{x}{y} = \log x - \log y$
4. $\log \left(\frac{1}{x}\right) = -\log x = \log x^{-1}$
5. $\log x^b = b \quad \log x$ $\begin{cases} \log 3^2 = 2\log 3 \\ \log sin^2 x = 2\log \sin x \\ \log \sin x^2 \neq 2\log \sin x \end{cases}$
6. $\log_a x = \frac{\ln x}{\ln a} \quad where\ a \neq 1$
7. $\log_{10} x = \frac{\ln x}{\ln 10}$
8. $\log 1 = 0$
9. $\log_a a = 1$

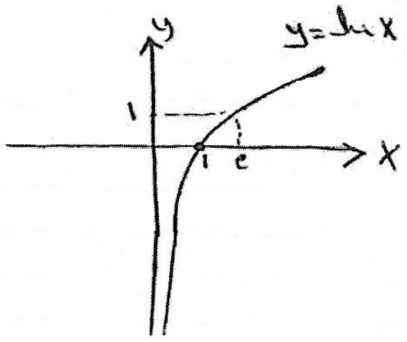
Graph of logarithmic function

ln

1) اذا كان الــ مقدار سالب فالنتيجة غير معرفة

2) اذا كان الــ x اقل من 1 فالناتج يكون سالب

و اذا كان الــ x اكبر من 1 فالناتج يكون موجب



Differentiations of logarithmic functions

$$\text{If } y = \log_a u(x) \ \ then \ \frac{dy}{dx} = \frac{1}{\ln a} * \frac{1}{u(x)} * \frac{d(u(x))}{dx}$$

Integration of logarithmic functions:-

$$\int \frac{u`(x)}{u(x)} = \ln|u(x)| + c$$

$$\int \frac{y`}{y} = \ln|y| + c$$

$$\int \frac{dx}{x} = \ln|x| + c$$

$$\int \frac{x\,dx}{x^2+1} = \frac{1}{2}\int \frac{2x\,dx}{x^2+1} = \frac{1}{2}\ln|x^2+1| + c$$

$$\int \frac{dx}{x\ln x} = \int \frac{\frac{1}{x}\,dx}{\ln x} = \ln|\ln x| + c$$

Find $\frac{dy}{dx}$:-

1) $y = \ln x^3 \quad \rightarrow y` = \frac{1}{x^3} * 3x^2 = \frac{3}{x}$

Or $y = 3\ln x \Rightarrow y` = 3 * \frac{1}{x}$

2) $y = \log_5(x^3 - x) \Rightarrow y` = \frac{1}{\ln 5} * \frac{1}{x^3 - x} * (3x^2 - 1)$

3) $y = \ln(\ln \sin x) \Rightarrow y` = \frac{1}{\ln(\sin x)} * \frac{1}{\sin x} * \cos x$

4) $y = x^{\cos x}$ (عندما الاس متغير نأخذ ln للطرفين)

$$\ln y = \ln x^{\cos x} \Rightarrow \ln y = \cos x \,.\ln x$$

$$\frac{1}{y} * y` = \cos x * \frac{1}{x} + \ln x * -\sin x$$

$$y` = x^{\cos x}\left[\frac{\cos x}{x} - \sin x \ln x\right]$$

5) $y = \sqrt[3]{\frac{(x+1)^2(x-3)^5}{(x^3+1)\cos x}} \Rightarrow y = \left[\frac{(x+1)^2(x-3)^5}{(x^3+1)\cos x}\right]^{\frac{1}{3}}$

$$\ln y = \ln\left[\frac{(x+1)^2(x-3)^5}{(x^3+1)\cos x}\right]^{\frac{1}{3}}$$

$$\ln y = \frac{1}{3}[\ln(x+1)^2(x-3)^5 - \ln(x^3+1)\cos x]$$

$$\ln y = \frac{1}{3}\ln(x+1)^2 + \frac{1}{3}\ln(x-3)^5 - \frac{1}{3}\ln(x^3+1) - \frac{1}{3}\ln\cos x$$

$$\ln y = \frac{2}{3}\ln(x+1) + \frac{5}{3}.\ln(x-3) - \frac{1}{3}\ln(x^3+1) - \frac{1}{3}\ln\cos x$$

$$\frac{y`}{y} = \frac{2}{3}\frac{1}{x+1} + \frac{5}{3}\frac{1}{x-3} - \frac{1}{3}\frac{3x^2}{x^3+1} - \frac{1}{3}\frac{1}{\cos x} * -\sin x$$

$$y` = \ldots\ldots\ldots..$$

6) $y = \sqrt[x]{\cos x} \Rightarrow y = (\cos x)^{\frac{1}{x}} \Rightarrow \ln y = \ln(\cos x)^{\frac{1}{x}}$

$$\ln y = \frac{1}{x}\ln\cos x$$

$$\frac{y`}{y} = \left[\frac{1}{x} * \frac{1}{\cos x} * -\sin x + \ln\cos x * -\frac{1}{x^2}\right]$$

$$y` = \sqrt[x]{\cos x}\left[\frac{1}{x} * \frac{1}{\cos x} * -\sin x + \ln\cos x * -\frac{1}{x^2}\right]$$

Evaluate the following integrals:

1) $\int \frac{\sin x\, dx}{3\cos x+5} = -\frac{1}{3}\int \frac{-3\sin x\, dx}{3\cos x+5} = -\frac{1}{3}\ln|3\cos x + 5| + c$

2) $\int \frac{\cos x\, dx}{\sqrt{\sin x}} = \int (\sin x)^{-\frac{1}{2}} \cos x\ \ dx = \frac{(\sin x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

3) $\int \frac{\cot x\, dx}{\log_5 \sin x} = \int \frac{\cot x\, dx}{\frac{\ln \sin x}{\ln 5}} = \ln 5 \int \frac{\cot x\, dx}{\ln \sin x} = \ln 5 \ln|\ln \sin x| + c$

4) $\int \frac{2x-5}{x} dx = \int \frac{2x}{x} dx - \int \frac{5}{x} dx = 2x - 5\ln|x| + c$

5) $\int \frac{dx}{x(\ln x)^3} = \int (\ln x)^{-3} . \frac{1}{x} dx = \frac{(\ln x)^{-2}}{-2} + c$

6) $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})} = 2\int \frac{\frac{1}{2}x^{-\frac{1}{2}}}{1+\sqrt{x}} = 2\ln\left|1 + \sqrt{x}\right| + c$

7) $\int \frac{dx}{\tan x(\ln \tan x)cos^2 x} = \int \frac{\frac{1}{\tan x} * sec^2 x\, dx}{\ln(\tan x)} = \ln|\ln \tan x| + c$

<u>**Ex:**</u> Find the values of (x) if

$2\ln\sqrt{x-1} + \ln(x+3) = \ln 5$

Solve: $\ln\left(\sqrt{x-1}\right)^2 + \ln(x+3) = \ln 5$

$$\ln(x-1)(x+3) = \ln 5$$

$$(x-1)(x+3) = 5$$

$$x^2 + 2x - 3 - 5 = 0 \Rightarrow x^2 + 2x - 8 = 0$$

$$(x-2)(x+4) = 0\ \ either\ x = 2\ or\ x = -4$$

Exponential function:

A) $y = e^x$ is called the exponential function, where (e) is the exponential number (e = 2.718)

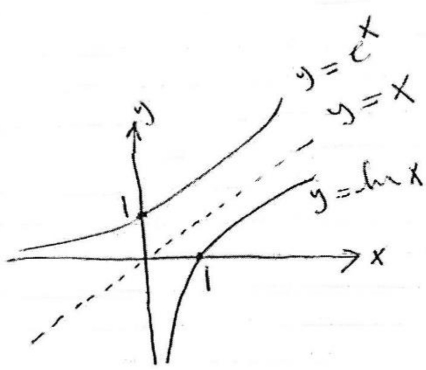
Notice that if $y = e^x$ then $\ln y = \ln e^x = x \ln e = x$

$$\ln e^x = e^{\ln x} = x$$

Properties of (e^x)

1) $e^{x_1+x_2} = e^{x_1}.e^{x_2}$
2) $e^0 = 1$
3) $e^{-x} = \frac{1}{e^x}$
4) $\ln e^x = x$
5) $e^{\ln x} = x$

Graph of $y = e^x$



Diff of exp. Function

If $y = e^{f(x)}$ $then$ $\frac{dy}{dx} = e^{f(x)}.f`(x)$
المشتقة = الاساس ln * الدالة الاسية * المشتقة للاس

Integration of Exp. function

Since $d(e^x) = e^x.dx$

$$\therefore \int e^x dx = e^x + c$$

Or in general

$$\boxed{\int e^u \left(\frac{du}{dx}\right) dx = e^u + c}$$

اذا توفرت المشتقة للاس في التكامل فالناتج هو الدالة الاصلية

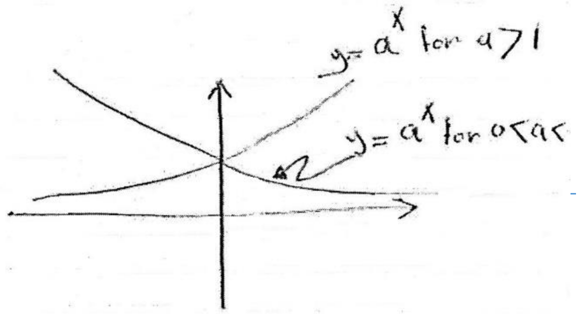
Ex: $\int e^{\sin x}.\cos x \; dx = e^{\sin x} + c$

B) $y = a^x$ is called the exponential function to base (a) where a > 0, a≠ 1

Notice that if $y = a^x$ then $y = e^{x \ln a}$

Graph of $y = a^x$

Differentiation of $y = a^x$



If $y = a^x$ where a > 0 a≠ 1

Then $\boxed{\frac{dy}{dx}\left(a^{f(x)}\right) = \ln a * a^{f(x)} * f`(x)}$

Integration of $y = a^x$

$$\boxed{\int a^{f(x)} . f`(x) = \frac{1}{\ln a} * a^{f(x)} + c}$$

Ex: Simplify the following:-

1) $e^{\ln(x^2-1)} = x^2 - 1 = (x-1)(x+1)$

2) $e^{\ln 3 + 2\ln 4} = e^{\ln 3 + \ln 4^2} = e^{\ln 3} * e^{\ln 4^2} = 3 * 4^2 = 48$

3) $\ln \frac{e^{3x}}{5} = \ln e^{3x} - \ln 5 = 3x \ln e - \ln 5 = 3x - \ln 5$

4) $\ln(x^2 * e^{-2x}) = \ln x^2 + \ln e^{-2x} = \ln x^2 - 2x = 2\ln x - 2x$

5) $e^{2x - \ln x} = \frac{e^{2x}}{e^{\ln x}} = \frac{e^{2x}}{x}$

Ex: Solve the following (find x):

1) $3^x = 2^{x+1} \Rightarrow \ln 3^x = \ln 2^{x+1} \Rightarrow x \ln 3 = (x+1)\ln 2$

$$x \ln 3 = x \ln 2 + \ln 2$$

$$x \ln 3 - x \ln 2 = \ln 2 \Rightarrow x(\ln 3 - \ln 2) = \ln 2$$

$$x \ln \frac{3}{2} = \ln 2 \Rightarrow x = \frac{\ln 2}{\ln \frac{3}{2}}$$

2) $x^x = 2^x \quad for\ x > 0$

$$x \ln x = x \ln 2 \Rightarrow x(\ln x - \ln 2) = 0$$

$$\therefore \ln x - \ln 2 = 0 \Rightarrow \ln x = \ln 2 \Rightarrow x = 2$$

3) $4^{-x} = 3^{x+2} \Rightarrow \quad \ln 4^{-x} = \ln 3^{x+2} \Rightarrow \quad -x \ln 4 = (x+2)\ln 3$

$$-x \ln 4 = x \ln 3 + 2 \ln 3$$

$$-x(\ln 4+\ln 3)=2\ln 3 \quad \Rightarrow \ x=-\frac{\ln 9}{\ln 12}$$

4) $\ln\left(x+\sqrt{x^2-1}\right)=1$

$$x+\sqrt{x^2-1}=e^1$$ نأخذ e للطرفين

$$\sqrt{x^2-1}=e-x$$ نربع للطرفين

$$x^2-1=e^2-2ex+x^2$$

$$2ex=e^2+1 \Rightarrow x=\frac{e^2+1}{2e}$$

5) $3^x+3^{-x}=4$ نضرب طرفي المعادلة 3^x

$$3^{2x}+1=4*3^x$$

$$3^{2x}-4*3^x+1=0$$

$$3^x=\frac{4\pm\sqrt{16-4}}{2} \Rightarrow 3^x=2\pm\sqrt{3}$$

$$x\ln 3=\ln\left(2\pm\sqrt{3}\right)$$

$$x=\frac{\ln\left(2\pm\sqrt{3}\right)}{\ln 3}$$

6) $\log_5(x+2)+\log_5(x-2)=1$

$$\log_5(x+2)(x-2)=1$$

$$5^1=(x+2)(x-2)$$

$$x^2-9=0 \Rightarrow x=\pm 3$$

Find $\frac{dy}{dx}$

1) $y=e^{\sin x} \Rightarrow y`=e^{\sin x}.\cos x$

2) $y = 2^{\csc x} \Rightarrow y` = 2^{\csc x} * \ln 2 * -\csc x \cot x$

3) $\tan y = e^x + \ln x$

$$sec^2y * y` = e^x + \frac{1}{x} \Rightarrow y` = \ldots\ldots.$$

4) $y = 2^x.3^x \Rightarrow \ln y = \ln(2^x.3^x)$

$$\ln y = \ln 2^x + \ln 3^x$$

$$\ln y = x\ln 2 + x\ln 3 = x(\ln 2 + \ln 3) = x\ln 6$$

$$\frac{y`}{y} = \ln 6 \Rightarrow y` = 2^x.3^x.\ln 6$$

5) $3^{1+e^y} = 2^{3+4^x}$

$$3^{1+e^y} * \ln 3 * e^y * \frac{dy}{dx} = 2^{3+4^x} * \ln 2 * 4^x * \ln 4$$

6) $y = e^{sin^3 e^{x^2}} \Rightarrow y` = e^{sin^3 e^{x^2}} * 3sin^2 e^{x^2} * \cos e^{x^2} * e^{x^2} * 2x$

7) $\ln yx = x^{y^2 \sin x} \Rightarrow \ln(\ln yx) = y^2 \sin x \ln x$

$$\frac{1}{\ln yx} * \frac{1}{yx} * (x.y` + y) = y^2 \sin x .\frac{1}{x} + \ln x\,(2yy` \sin x + y^2 \cos x)$$

8) $y = x^{\sin x} + x^2$ لا يمكن اخذ ln الطرفين مباشرة وذلك لكون الطرف الايمن ليس دالة منفصلة

$$y = e^{\ln x^{\sin x}} + x^2$$

$$y = e^{\sin x \ln x} + x^2$$

$$y` = e^{\sin x \ln x}\left(sinx * \frac{1}{x} + cosx.lnx\right) + 2x$$

$$\therefore y` = x^{\sin x}\left(sinx * \frac{1}{x} + cosx.lnx\right) + 2x$$

9) $y = (x^2+1)^{x^3} + \sin(x^x) + x^4$

$$y = e^{\ln(x^2+1)^{x^3}} + \sin\left(e^{\ln x^x}\right) + x^4$$

$$= e^{x^3\ln(x^2+1)} + \sin\left(e^{x\ln x}\right) + x^4$$

$$y` = e^{x^3\ln(x^2+1)}\left[x^3 * \frac{2x}{x^2+1} + \ln(x^2+1) * 3x^2\right]$$

$$+ \cos\left(e^{x\ln x}\right)\left[e^{x\ln x}\left(x.\frac{1}{x} + \ln x\right)\right] + 4x^3$$

10) $y = \tan x^x \Rightarrow y = \tan e^{\ln x^x} = \tan e^{x\ln x}$

$$y` = sec^2\left(e^{x\ln x}\right) * e^{x\ln x} * \left(x.\frac{1}{x} + \ln x\right)$$

<u>Ex:</u> Evaluate the following integrals:-

1) $\int (e^x+2)\,dx = \int e^x\,dx + 2\int dx = e^x + 2x + c$

2) $\int \frac{e^{2x}\,dx}{3e^{2x}+1} = \frac{1}{6}\int \frac{6e^{2x}\,dx}{3e^{2x}+1} = \frac{1}{6}\ln|3e^{2x}+1| + c$

3) $\int e^{\ln\sqrt{x+1}}\;dx = \int \sqrt{x+1}\;dx = \frac{(x+1)^{3/2}}{3/2} + c$

4) $\int 3^{\tan 7x}.sec^2 7x\,dx = \frac{1}{7\ln 3}3^{\tan 7x} + c$

5) $\int 2^x.\cos 2^x\,dx = \frac{1}{\ln 2}\int 2^x * \ln 2 * \cos 2^x\;dx \;= \frac{1}{\ln 2} * \sin 2^x + c$

6) $\int x^2\cot(2+x^3)\,dx = \int x^2\frac{\cos(2+x^3)}{\sin(2+x^3)}dx = \frac{1}{3}\ln|\sin(2+x^3)| + c$

7) $\int e^{2x}sec^2(e^{2x}+1)\,dx = \frac{1}{2}\tan(e^{2x}+1) + c$

8) $\int \frac{\ln x}{x}dx = \frac{(\ln|x|)^2}{2} + c$

9) $\int \frac{\ln(\sin x)}{\tan x}dx = \int (\ln\sin x)\frac{\cos x}{\sin x}dx = \frac{1}{2}(\ln(\sin x))^2 + c$

10) $\int \frac{(x-1)\,dx}{3x^2-6x+5} = \frac{1}{6}\ln|3x^2-6x+5| + c$

11) $\int \frac{sec^3x+e^{\sin x}}{\sec x} dx = \int \frac{sec^3x}{\sec x} dx + \int \frac{e^{\sin x}}{\sec x} dx$

$$= \int sec^2x\, dx + \int e^{\sin x} * \cos x \;\, dx \; = \tan x + e^{\sin x} + c$$

12) $\int_1^{e^2} \frac{2\ln x}{x} dx = \frac{2(\ln x)^2}{2}\Big|_1^{e^2} = (\ln e^2)^2 - (\ln 1)^2 = (2\ln e)^2 = 4 - 0 = 4$

13) $\int e^{sin^2x}.\sin 2x\, dx = \int e^{sin^2x} * 2\sin x\cos x \; = e^{sin^2x} + c$

14) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2e^{\sqrt{x}} + c$

6 Inverse trigonometric functions

The inverse trigonometric functions are defined as follows

$$y = sin^{-1}x \Rightarrow x = \sin y$$

$$y = cos^{-1}x \Rightarrow x = \cos y$$

$$y = tan^{-1}x \Rightarrow x = \tan y$$

$$y = sec^{-1}x \Rightarrow x = \sec y$$

$$y = cot^{-1}x \Rightarrow x = \cot y$$

$$y = csc^{-1}x \Rightarrow x = \csc y$$

Ex: If $sin^{-1}\frac{1}{2} = x \;\therefore \sin x = \frac{1}{2} \;\; but \;\; \sin\frac{\pi}{6} = \frac{1}{2} \quad \Rightarrow \; x = \frac{\pi}{6}$

Inverse trigonometric identities:-

1) $sin^{-1}x + cos^{-1}x = \frac{\pi}{2}$

2) $sin^{-1}(\sin\theta) = \theta$

3) $\sin(sin^{-1}\theta) = \theta$

4) $cos^{-1}(-x) = \pi - cos^{-1}(x)$

5) $sec^{-1}(x) = cos^{-1}\left(\frac{1}{x}\right)$

6) $csc^{-1}(x) = sin^{-1}\left(\frac{1}{x}\right)$

7) $sec^{-1}(-x) = \pi - sec^{-1}(x)$

Derivative inverse trigonometric function:-

1) $d(sin^{-1}u) = \frac{du}{\sqrt{1-u^2}}$

2) $d(cos^{-1}u) = -\frac{du}{\sqrt{1-u^2}}$

3) $d(tan^{-1}u) = \frac{du}{1+u^2}$

4) $d(cot^{-1}u) = -\frac{du}{1+u^2}$

5) $d(sec^{-1}u) = \frac{du}{|u|\sqrt{u^2-1}}$

6) $d(csc^{-1}u) = -\frac{du}{|u|\sqrt{u^2-1}}$

Integration leading to inverse trigonometric functions

$$\int \frac{dx}{\sqrt{1-x^2}} = sin^{-1}x + c$$

$$\int \frac{dx}{1+x^2} = tan^{-1}x + c$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = sec^{-1}x + c$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} tan^{-1}\left(\frac{x}{a}\right) + c$$

Examples

A) Show that

1) $sin^{-1}(x)$ is an odd function

Let $y = sin^{-1}(-x) \Rightarrow -x = \sin y$

$$-y = sin^{-1}x \Rightarrow y = -sin^{-1}x$$

$\therefore sin^{-1}(-x) = -sin^{-1}(x)$ odd function

B) Evaluate $\cos(sin^{-1}0.8)$

$$\cos A = \sqrt{1 - sin^2 A}$$

$$\cos(sin^{-1}0.8) = \sqrt{1 - sin^2(sin^{-1}0.8)}$$

$$= \sqrt{1 - \sin(sin^{-1}0.8).\sin(sin^{-1}0.8)}$$

$$= \sqrt{1-(0.8)(0.8)} = \sqrt{1-0.64} = 0.6$$

C) Find $\frac{dy}{dx}$

1) $y = sin^{-1}(x^2) \Rightarrow y\grave{} = \frac{2x}{\sqrt{1-x^4}}$

2) $y = cot^{-1}(x \sin x) \Rightarrow y\grave{} = \frac{-(x\cos x + \sin x)}{1+(x \sin x)^2}$

3) $sec^{-1}(x\sqrt{y}) = sin^{-1}\left(\frac{3}{\ln x}\right)$

$$\frac{x.\frac{y\grave{}}{2\sqrt{y}} + \sqrt{y}}{|x\sqrt{y}|\sqrt{x^2y-1}} = \frac{\frac{-3 * \frac{1}{x}}{(\ln x)^2}}{\sqrt{1-\left(\frac{3}{\ln x}\right)^2}}$$

4) $sin^{-1}(x^y) + (cos^{-1}y)^{sin^{-1}x} = x^2$

$sin^{-1}\left(e^{\ln x^y}\right) + e^{\ln(cos^{-1}y)^{sin^{-1}x}} = x^2$

$sin^{-1}\left(e^{y \ln x}\right) + e^{sin^{-1}x \ln(cos^{-1}y)} = x^2$

$$\Rightarrow \frac{e^{y\ln x}\left(\frac{y}{x}+\ln x y`\right)}{\sqrt{1-(e^{y\ln x})^2}}$$

$$+ e^{sin^{-1}x\ln(cos^{-1}y)}\left[sin^{-1}x\frac{1}{cos^{-1}y} * \frac{-y`}{\sqrt{1-y^2}}\right.$$

$$\left. + \ln(cos^{-1}y)\frac{1}{\sqrt{1-x^2}}\right] = 2x$$

D) Evaluate the following integrals:-

1) $\int \frac{dx}{\sqrt{4-x^2}} = sin^{-1}\frac{x}{2} + c$ $\quad note: \left[\begin{array}{l} f = x \\ df = dx \\ a = 2 \end{array}\right.$

2) $\int \frac{dx}{x\sqrt{1-ln^2 x}} = sin^{-1}(\ln x) + c$ $\quad note: \left[\begin{array}{l} f = \ln x \\ df = \frac{1}{x}dx \\ a = 1 \end{array}\right.$

3) $\int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int (1-x^2)^{-\frac{1}{2}} * -2x\,dx = -\frac{1}{2}\frac{\sqrt{1-x^2}}{1/2} = -\sqrt{1-x^2} + c$

4) $\int \frac{sin^{-1}x\,dx}{\sqrt{1-x^2}} = \frac{(sin^{-1})^2}{2} + c$

5) $\int \frac{2^x\,dx}{1+4^x} = \int \frac{2^x\,dx}{1+(2^x)^2} = \frac{1}{\ln 2}tan^{-1}2^x + c$ $\quad Note: \left[\begin{array}{l} f = 2^x \\ df = 2^x\ln 2\,dx \\ a = 1 \end{array}\right.$

6) $\int \frac{x^2\,dx}{\sqrt{1-x^6}} = \frac{1}{3}\int \frac{3x^2\,dx}{\sqrt{1-(x^3)^2}} = \frac{1}{3}sin^{-1}(x^3) + c$ $\quad Note: \left[\begin{array}{l} f = x^3 \\ df = 3x^2\,dx \\ a = 1 \end{array}\right.$

7) $\int \frac{dx}{3+2x^2} = \frac{1}{2}\int \frac{dx}{\frac{3}{2}+x^2} = \frac{1}{2}\frac{1}{\sqrt{\frac{3}{2}}}tan^{-1}\left(\frac{x}{\sqrt{\frac{3}{2}}}\right) + c$ $\quad note: \left[\begin{array}{l} f = x \\ df = dx \\ a = \sqrt{\frac{3}{2}} \end{array}\right.$

8) $\int \frac{(1+x)\,dx}{4+x^2} = \int \frac{dx}{4+x^2} + \frac{1}{2}\int \frac{2x\,dx}{4+x^2}$

$$= \frac{1}{2} tan^{-1}\frac{x}{2} + \frac{1}{2}\ln|4 + x^2| + C$$

9) $\int \frac{dx}{x^2+4x+8} = \int \frac{dx}{x^2+4x+4-4+8} = \int \frac{dx}{(x+2)^2+4}$ $note: \begin{bmatrix} f = x+2 \\ df = dx \\ a = 2 \end{bmatrix}$

$$= \frac{1}{2} tan^{-1}\left(\frac{x+2}{2}\right) + c$$

10) $\int \frac{dx}{\sqrt{6x-x^2-8}} = \int \frac{dx}{\sqrt{-(x^2-6x+8)}} = \int \frac{dx}{\sqrt{-(x^2-6x+9-9+8)}}$

$$= \int \frac{dx}{\sqrt{-[(x-3)^2-1]}} = \int \frac{dx}{\sqrt{1-(x-3)^2}}$$ $note: \begin{bmatrix} f = x-3 \\ af = dx \\ a = 1 \end{bmatrix}$

$$= sin^{-1}(x-3) + c$$

11) $\int \frac{\cos x\, dx}{\sqrt{1-sin^2 x}} = sin^{-1}(\sin x) = x + c$

12) $\int \frac{\left(tan^{-1}\sqrt{x}\right)^3}{\sqrt{x}(1+x)} = 2\int \frac{\left(tan^{-1}\sqrt{x}\right)^3}{2\sqrt{x}(1+x)} = \frac{2\left(tan^{-1}\sqrt{x}\right)^4}{4} + c$

13) $\int \frac{3x^3-4x^2+3x}{x^2+1} dx$ ملاحظة: اذا كانت الدالة كسرية واس البسط اكبر او يساوي اس المقام نقوم بتقسيم البسط على المقام اولا .

$$= \int (3x-4) + \frac{4}{x^2+1}\, dx$$

$$= \int 3x\, dx - 4\int dx + 4\int \frac{1}{x^2+1}\, dx$$

$$= \frac{3x^2}{2} - 4x + 4tan^{-1}(x) + c$$

$$\begin{array}{r|l} & 3x - 4 \\ \hline x^2+1 & 3x^3 - 4x^2 + 3x \\ & 3x^3 + 3x \\ \hline & -4x^2 \\ & -4x^2 - 4 \\ \hline & +4 \end{array}$$